Systematic Risk in Homogeneous Credit Portfolios*

Christian Bluhm† and Ludger Overbeck **§

1 Systematic Risk in Credit Portfolios

In credit portfolios (see [5] for an introduction) there are typically two types of counterparties: Listed firms and non-listed borrowers. For the first type, a time series of the firm's equity values can be used to derive an *Ability-to-Pay Process* (APP), showing for every point in time the firm's ability to pay, see e.g. [6]. For the second type, equity processes are not available, but still every borrower somehow admits an APP, depending on the customer's assets and liabilities, sometimes known by the lending institute, but in any case imposed as an unobservable *latent variable*. In general, we can expect that correlations between the obligor's APPs strongly influence the portfolio's credit risk. The calculation of *APP correlations* usually is based on a suitable factor model, e.g., a (single-beta) linear model

$$r_i \;=\; \beta_i \Phi_i + \varepsilon_i \; , \tag{1}$$

where r_i denotes the standardized log-return of the i-th borrower's APP, Φ_i denotes the *composite factor* of borrower i, and ε_i denotes the *residual* part of r_i, which can not be explained by the customer's composite factor. Usually the composite factor of a borrower is itself a weighted sum of country- and industry-related indices, see e.g. [5], Chapter 1. Along with representation (1) comes a decomposition of variance,

$$\mathbb{V}[r_i] \;=\; 1 \;=\; \underbrace{\beta_i^2 \mathbb{V}[\Phi_i]}_{=R_i^2,\text{ systematic}} + \underbrace{\mathbb{V}[\varepsilon_i]}_{=1\text{-}R_i^2,\text{ specific}} \tag{2}$$

in a *systematic* and an *idiosyncratic* effect. The systematic part of variance is the so-called *coefficient of determination*, denoted by R_i^2, implicitly determined by the regression (1). It can be seen as a quantification of the *systematic risk* of borrower i and is an important input parameter in credit portfolio management tools, heavily driving the portfolio's *Economic Capital*[1] (EC). For example, the following chart shows CEC, the *contributory EC* (w.r.t. a reference portfolio of corporate loans to middle-size companies) as a function of R^2 for a loan with a *default probability* of 30bps, a *severity*[2] of 50%, a 100% country weight in Germany, and a 100% industry weight in automotive industry:

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†HypoVereinsbank (Munich)

**Deutsche Bank (Frankfurt)

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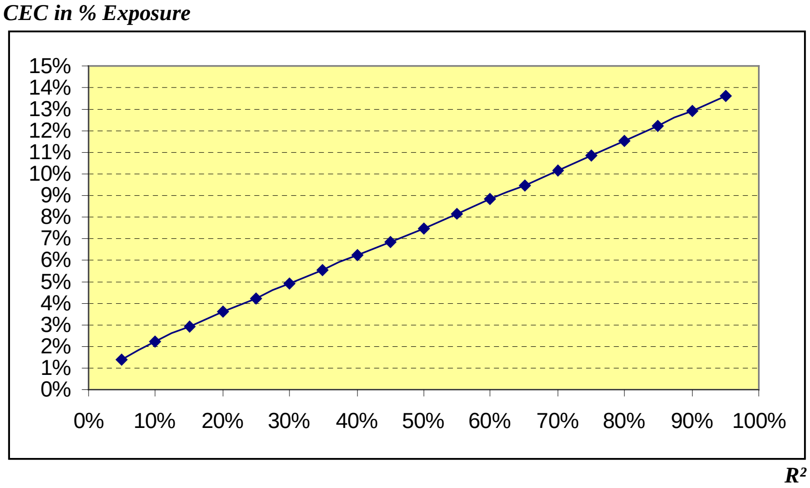
[1]The Economic Capital w.r.t. a level of confidence α of a credit portfolio is defined as the α-quantile of the portfolio's loss distribution minus the portfolio's expected loss (i.e. the mean of the portfolio's loss distribution).

[2]A severity of 50% means that in case of default the recovery rate can be expected to be (1-severity)=50%.



The chart shows that the increase in contributory EC implied by an increase of systematic risk (quantified by R^2) is significant.

This note has a two-fold intention: First, we want to present a simple approach for estimating the systematic risk, that is, the parameter R^2, for a homogeneous credit portfolio. Second, we discuss the proposal of the *Basel Committee on Banking Supervision*, see [1], to fix the *asset correlation* respectively the systematic risk for the calibration of the benchmark risk weights for corporate loans at the 20% level. In our discussion we apply the method introduced in the first part of this note to Moody's corporate bond default statistics and compare our estimated APP correlation with the correlation level suggested in the Basel II consultative document. On one hand, our findings show that the average asset correlation within a rating class is close to the one suggested by the Basel II approach. On the other hand, the assumption of a *one-factor model* with a uniform asset correlation of 20% as suggested in the current draft of the new capital accord turns out to be violated as soon as we consider the correlation between different segments, e.g., rating segments as in our case.

2 Homogeneous Credit Portfolios

The simplest way to model default or survival of borrowers in a credit portfolio is by means of binary random variables, where a '1' indicates default and a '0' means survival w.r.t. to a certain valuation horizon.

2.1 A General Mixture Model for Uniform Portfolios

We start with a standard mixture model of *exchangeable binary random variables*, see [10], 7.1.3. More precisely, we model our credit portfolio by a sequence of Bernoulli random variables $L_1, ..., L_m \sim B(1; p)$, where the default probability p is random with a distribution function F (with support in $[0, 1]$), such that given p, the variables $L_1, ..., L_m$ are conditionally i.i.d. The (unconditional) joint distribution of the L_i's is then determined by the probabilities

$$\mathbb{P}[L_1 = l_1, ..., L_m = l_m] \;=\; \int_0^1 p^k(1-p)^{m-k} dF(p), \qquad k = \sum_{i=1}^{m} l_i, \qquad l_i \in \{0, 1\}. \tag{3}$$

The uniform default probability of borrowers in the portfolio is given by

$$\overline{p} = \mathbb{P}[L_i = 1] = \int_0^1 p\, dF(p) \tag{4}$$

and the uniform default correlation between different counterparties is

$$r = \text{Corr}(L_i, L_j) = \frac{\mathbb{P}[L_i = 1, L_j = 1] - \overline{p}^2}{\overline{p}(1 - \overline{p})} = \frac{\int_0^1 p^2 dF(p) - \overline{p}^2}{\overline{p}(1 - \overline{p})} \,. \tag{5}$$

Therefore, $\text{Corr}(L_i, L_j) = \mathbb{V}[Z]/(\overline{p}(1 - \overline{p}))$, where Z is a random variable with distribution F, showing that the dependence between the L_i's is either positive or zero. Moreover, $\text{Corr}(L_i, L_j) = 0$ is only possible if F is a Dirac distribution (degenerate case). The other extreme, $\text{Corr}(L_i, L_j) = 1$, can only occur if F is a Bernoulli distribution, $F \sim B(1; \overline{p})$.

2.2 Construction of a Homogeneous Portfolio

Being started from a general perspective, we now briefly elaborate one possible approach to construct a mixture distribution F reflecting the APP-model indicated in the introduction. Following the classical *Asset Value Model* of MERTON [11] and BLACK / SCHOLES [4], we model the borrower's APPs as correlated geometric Brownian motions,

$$dA_t(i) = \mu_i A_t(i) dt + \sigma_i A_t(i) dB_t(i) \qquad (i = 1, ..., m), \tag{6}$$

where $(B_t(1), ..., B_t(m))_{t \geq 0}$ is a multivariate Brownian motion with correlation ϱ (the *uniform APP-correlation*). Assuming a one-year time window, the vector of asset returns at the valuation horizon, $\left(\ln \frac{A_1(1)}{A_0(1)}\ , ... \, ,\ \ln \frac{A_1(m)}{A_0(m)}\right)$, is multivariate normal with mean vector $(\mu_1 - 0.5\,\sigma^2, ..., \mu_m - 0.5\,\sigma^2)$ and covariance matrix $\Sigma = (\sigma_i \sigma_j \varrho_{ij})_{1 \leq i,j \leq m}$ where $\varrho_{ij} = \varrho$ if $i \neq j$ and $\varrho_{ij} = 1$ if $i = j$. A standard assumption in this context is the existence of a so-called *default point* $\tilde{c}_i$ for every borrower i such that i defaults if and only if its APP at the valuation horizon falls below $\tilde{c}_i$, see CROSBIE [6] for more information about the calibration of default points. So we can define binary variables by a *latent variables* approach,

$$L_i = 1 \quad \Longleftrightarrow \quad A_1(i) < \tilde{c}_i \qquad (i = 1, ..., m).$$

As a consequence of the chosen framework we obtain

$$\overline{p} = \mathbb{P}[L_i = 1] = \mathbb{P}[A_1(i) < \tilde{c}_i] = \mathbb{P}[X_i < c_i] = N[c_i], \tag{7}$$

where N denotes the standard normal distribution function, the variables X_i are standard normal with uniform correlation ϱ, and $c_i = (\ln \tilde{c}_i - \ln A_0(i) - \mu_i + 0.5\,\sigma_i^2)/\sigma_i$. Moreover, (7) shows that the c_i's must be equal to a constant c, namely the $\overline{p}$-quantile of the standard normal distribution, $c = N^{-1}(\overline{p})$. Because the distribution of a Gaussian vector is uniquely determined by their expectation vector and covariance matrix, we can parametrize the variables X_i by means of a *one-factor model*

$$X_i \;=\; \underbrace{\sqrt{\varrho}\, Y}_{\text{systematic}} + \underbrace{\sqrt{1 - \varrho}\, Z_i}_{\text{specific}} \qquad (i = 1, ..., m), \tag{8}$$

where $Y, Z_1, ..., Z_m$ are independent standard normal random variables. Equation (8) is obviously a linear regression equation, and based on (1) and (2) we see that the systematic risk or R^2 of the regression is given by the APP-correlation ϱ.
Therefore, estimating systematic risk within our parametric framework means estimating the asset respectively APP correlation ϱ. As soon as ϱ is determined, the *default correlation* r is also known, because based on equation (5) we only need to know the joint default probability $\mathbb{P}[L_i = 1, L_j = 1]$. Because the X_i's are standard normal, the *Joint Default Probability* (JDP) is given by the bivariate normal integral

$$\mathbb{P}[X_i < c, X_j < c] \;=\; \frac{1}{2\pi\sqrt{1-\varrho^2}} \int_{-\infty}^{N^{-1}(\overline{p})} \int_{-\infty}^{N^{-1}(\overline{p})} e^{-\frac{1}{2}(x_i^2 - 2\varrho x_1 x_2 + x_2^2)/(1-\varrho^2)} dx_1 dx_2 \,. \tag{9}$$

So for fixed $\overline{p}$ we can derive r from ϱ and vice versa by evaluating formulas (5) and (9).

At this point we come back to the distribution F in our mixture model (3). From (8) we derive

$$\overline{p} = \mathbb{P}[L_i = 1] = \int_{-\infty}^{\infty} \mathbb{P}[L_i = 1 \mid Y = y]\, dN(y) = \int_{-\infty}^{\infty} g(y)\, dN(y) \qquad \text{where}$$

$$\begin{aligned} g(y) = \mathbb{P}[L_i = 1 \mid Y = y] &= \mathbb{P}\Big[\sqrt{\varrho}\, Y + \sqrt{1-\varrho}\, Z_i < c_i \mid Y = y\Big] \\ &= \mathbb{P}\Big[Z_i < \frac{c - \sqrt{\varrho}\, Y}{\sqrt{1-\varrho}} \mid Y = y\Big] = N\Big[\frac{N^{-1}(\overline{p}) - \sqrt{\varrho}\, y}{\sqrt{1-\varrho}}\Big] \,, \end{aligned} \tag{10}$$

because Z_i is standard normal. We therefore obtain equation (4) with F being the distribution function of the random variable $g(Y)$, $Y \sim N(0, 1)$,

$$F \;=\; N(0,1) \circ g^{-1} \,. \tag{11}$$

Note that this is just one possible approach to realize a mixture model of exchangeable binary variables. The fundamental assumption here is the log-normality of APPs. For related work regarding homogeneous or uniform portfolios we refer to BELKIN ET. AL. [2]-[3], FINGER [9], and VASICEK [13]. For a more detailed investigation of mixture models applied to credit risk modelling we refer to FREY AND MCNEIL [8], and to Chapter 2 in [5].

3 Estimation of Correlation

In this section we fix F as in (11) and assume the underlying model. The (percentage) portfolio loss is given by $L = \frac{1}{m}\sum_{i=1}^{m} L_i$, and its distribution is determined by (3). We assume that we observed a time-series of vectors of default events $(\hat{L}_{j1}, ..., \hat{L}_{jm_j})_{j=1,...,n}$ where j refers to the year of observation and m_j denotes the number of counterparties in the portfolio in year j. The write-offs immediately imply default frequencies

$$\hat{p}_j = \frac{1}{m_j}\sum_{i=1}^{m_j} \hat{L}_{ji} \qquad (j = 1, ..., n).$$

According to our model assumption and Equation (10) we can also write

$$g(y_j) = \hat{p}_j = \frac{1}{m_j} \sum_{i=1}^{m_j} \hat{L}_{ji} \; ,$$

where y_j denotes the (unknown!) realization of the factor Y in year j. Conditional on y_j the variables L_{ji} are i.i.d. Bernoulli for fixed j. The observed default frequency $\hat{p}_j$ therefore constitutes the standard maximum-likelihood estimate for the default probability $g(y_j)$ of year j. Assuming $y_1, ..., y_n$ to be realizations of i.i.d. copies $Y_1, ..., Y_n$ of Y, we obtain

$$\frac{1}{n} \sum_{j=1}^{n} g(Y_j) \quad \stackrel{n\to\infty}{\longrightarrow} \quad \mathbb{E}[g(Y)] \; = \; \overline{p} \qquad \text{a.s.}$$

$$\frac{1}{n-1} \sum_{j=1}^{n} \Big(g(Y_j) - \overline{g(Y)} \Big)^2 \quad \stackrel{n\to\infty}{\longrightarrow} \quad \mathbb{V}[g(Y)] \qquad \text{a.s.} \tag{12}$$

where $\overline{g(Y)} = \sum g(Y_j)/n$. Therefore, the sample mean and variance

$$m_p = \frac{1}{n} \sum_{j=1}^{n} \hat{p}_j \qquad \text{and} \qquad s_p^2 = \frac{1}{n-1} \sum_{j=1}^{n} (\hat{p}_j - m_p)^2$$

are reasonable estimates of the mean and variance of $g(Y)$. The underlying unknown asset correlation ϱ is the only 'free' parameter in the variance of $g(Y)$. Then, (5) and (9) yield

$$\mathbb{V}\left[g(Y)\right] = \mathbb{E}[g(Y)^2] - \mathbb{E}[g(Y)]^2 = \int_0^1 p^2 dF(p) - \overline{p}^2 \; =$$

$$= \; \frac{1}{2\pi\sqrt{1-\varrho^2}} \int_{-\infty}^{N^{-1}(\overline{p})} \int_{-\infty}^{N^{-1}(\overline{p})} e^{-\frac{1}{2}(x_i^2 - 2\varrho x_1 x_2 + x_2^2)/(1-\varrho^2)} dx_1 dx_2 \; - \; \overline{p}^2 \; . \tag{13}$$

Estimating $\mathbb{V}\left[g(Y)\right]$ by s_p^2 and $\overline{p}$ by m_p, we can now determine ϱ by solving the equation

$$s_p^2 \; = \; N_2\left[N^{-1}(m_p), N^{-1}(m_p); \varrho\right] - m_p^2 \tag{14}$$

for ϱ. Here, $N_2(\cdot, \cdot, \rho)$ denotes the standard bivariate normal distribution function with correlation ρ. Note again that in (14) only ϱ is unknown.
Equation (14) represents a very simple method for estimating asset respectively APP correlations for homogeneous credit portfolios. Because in many cases portfolios admit an *analytical approximation* by a suitably calibrated uniform portfolio, *systematic risk* can be estimated for portfolios admitting a representation by a synthetic homogeneous reference portfolio.

4 Beyond Models with Uniform R-Squared

By a similar approach we can derive asset/APP correlations between different *segments* (e.g., rating classes; see the next sections) from the time series of default rates in the considered segments.

4.1 Correlation Between Segments - Basic Version

This section presents a straightforward application of Equation (14), interpreted in a slightly different manner. In our example we define two segments: Segment 1 consists of Moody's universe of Baa-rated corporate bonds, whereas Segment 2 consists of Ba-rated bonds. The idea now is to pick a 'typical' bond from every segment and to calculate the asset correlation ϱ between these bonds. Because segments are assumed to behave like a uniform portfolio, the so calculated correlation must be equal to the correlation between the segments.
More explicitely, we proceed as follows. Denote the covariance of the default event of an obligor in class Baa and an obligor in class Ba by $\text{Cov}_{Baa,Ba}$. Our model assumptions yield

$$\text{Cov}_{Baa,Ba} = \mathbb{P}[L_{Baa,i} = 1, L_{Ba,j} = 1] - \overline{p}_{Baa}\overline{p}_{Ba} = \tag{15}$$

$$= N_2\left[N^{-1}(\overline{p}_{Baa}), N^{-1}(\overline{p}_{Ba}); \varrho\right] - \overline{p}_{Baa}\overline{p}_{Ba}\ .$$

Here, $L_{Baa,i}$ and $L_{Ba,j}$ are loss variables referring to bonds in rating class Baa and Ba. The parameters $\overline{p}_{Baa}$ and $\overline{p}_{Ba}$ are the corresponding default probabilities. By a result similar to Equation (12) we can estimate this covariance by the sample covariance of the time series of default rates, see also Equation (20). Replacing the default probabilities by the corresponding sample means and solving (15) for ϱ yields the correlation between rating classes Baa and Ba.

4.2 Correlation Between Segments - Multi Index Approach

In this section we follow a slightly more complex approach. Let us assume that we have m different segments, for example, rating classes or industry buckets. Every segment k will be considered as a uniform portfolio with default probability p_k and asset correlation ϱ_k. Equation (8) can then be rewritten by

$$X_{ki} \;=\; \sqrt{\varrho_k}\, Y_k \;+\; \sqrt{1-\varrho_k}\, Z_{ki} \qquad (k = 1, ..., m;\ i = 1, ..., m_k), \tag{16}$$

where Y_k denotes a segment-specific index, and Z_{ki} is the specific effect of obligor i in segment k. The number of obligors in segment k is given by m_k. Additionally we introduce a global factor Y by means of which all segments are correlated,

$$Y_k \;=\; \sqrt{\varrho}\, Y \;+\; \sqrt{1-\varrho}\, Z_k \qquad (k = 1, ..., m), \tag{17}$$

where ϱ is the uniform R^2 of the segment indices w.r.t. the global factor Y. The variables Z_k are the segment-specific effects. It is assumed that the variables Y, Z_k, Z_{ki} are independent standard normal random variables. The correlation ϱ is the unknown quantity we want to determine in the sequel; see Equation (20).
To give an example, let us consider the two extreme cases regarding ϱ. In case of $\varrho = 0$, the segments are uncorrelated. In case of $\varrho = 1$, the segments are perfectly correlated, such that the union of the segments yields an aggregated uniform portfolio. In both cases, the R^2 of obligors depends on the obligor's segment k and is given by ϱ_k. The correlation matrix $C = (c_{\sigma\tau})_{1\le\sigma,\tau\le m_1+...+m_m}$ of the portfolio consisting of the union of all segments is given by

$$c_{\sigma\tau} \;=\; \text{Corr}[X_{k_\sigma i_\sigma}, X_{k_\tau i_\tau}] \;=\; \sqrt{\varrho_{k_\sigma}\varrho_{k_\tau}}\,\varrho \;+\; \sqrt{\varrho_{k_\sigma}\varrho_{k_\tau}}\,(1-\varrho)\,\text{Corr}[Z_{k_\sigma}Z_{k_\tau}] \;+ \tag{18}$$

$$+\sqrt{(1-\varrho_{k_\sigma})(1-\varrho_{k_\tau})}\,\text{Corr}[Z_{k_\sigma i_\sigma} Z_{k_\tau i_\tau}] \;=\; \begin{cases} \varrho_k & \text{if } \; k_\sigma = k_\tau = k,\; i_\sigma \neq i_\tau \\ 1 & \text{if } \; k_\sigma = k_\tau = k,\; i_\sigma = i_\tau \\ \sqrt{\varrho_{k_\sigma}\varrho_{k_\tau}}\,\varrho & \text{if } \; k_\sigma \neq k_\tau \end{cases} .$$

Equation (18) confirms ϱ_k as a *segment intra-correlation*, whereas the correlation between counterparties from different segments k_σ and k_τ is given by $\sqrt{\varrho_{k_\sigma}\varrho_{k_\tau}}\,\varrho$.
By arguments analogous to the one in Section 3, one can see that the empirical covariance of the default rates of different segments over time converges against the theoretical covariance

$$\text{Cov}[p_{k_\sigma}(Y_{k_\sigma}), p_{k_\tau}(Y_{k_\tau})] \;=\; \int_{\mathbb{R}^2} p_{k_\sigma}(y_{k_\sigma}) p_{k_\tau}(y_{k_\tau})\, dN_2(y_{k_\sigma}, y_{k_\tau}|\varrho) \;-\; \overline{p}_{k_\sigma}\overline{p}_{k_\tau} \;, \tag{19}$$

where the functions $p_k(\cdot)$, $k = 1, ..., m$, are defined by

$$p_k(y_k) = N\Big[\frac{N^{-1}(\overline{p}_k) - \sqrt{\varrho_k}\, y_k}{\sqrt{1-\varrho_k}}\Big] \;,$$

reflecting the same arguments as presented in (10).
Comparing the empirical with the theoretical covariance, we obtain the following Equation, where n refers to the number of considered years:

$$\frac{1}{2\pi\sqrt{1-\varrho^2}} \int_{\mathbb{R}}\int_{\mathbb{R}} N\Big[\frac{N^{-1}(\overline{p}_{k_\sigma}) - \sqrt{\varrho_{k_\sigma}}\, y_{k_\sigma}}{\sqrt{1-\varrho_{k_\sigma}}}\Big]\, N\Big[\frac{N^{-1}(\overline{p}_{k_\tau}) - \sqrt{\varrho_{k_\tau}}\, y_{k_\tau}}{\sqrt{1-\varrho_{k_\tau}}}\Big] \times \tag{20}$$

$$\times\; e^{-\frac{1}{2(1-\varrho^2)}(y_{k_\sigma}^2 - 2\varrho\, y_{k_\sigma} y_{k_\tau} + y_{k_\tau}^2)} dy_{k_\sigma}\, dy_{k_\tau} \;-\; \overline{p}_{k_\sigma}\overline{p}_{k_\tau} \;\overset{!}{=}\; \frac{1}{n}\sum_{j=1}^{n} \big(p_{k_\sigma j} - \overline{p}_{k_\sigma}\big)\big(p_{k_\tau j} - \overline{p}_{k_\tau}\big) \;,$$

where p_{kj} denotes the default frequency of segment k in year j. Replacing the $\overline{p}_k$'s by sample means, the only unknown parameter in Equation (20) is the correlation ϱ between segments k_σ and k_τ. Therefore, we can solve (20) in order to get an estimate for ϱ.

5 The 20% Correlation Assumption of Basel II

As already mentioned in the introduction, the new Basel capital accord in its recent version suggests a 20% -level of systematic risk for the calibration of the benchmark risk weights for corporate loans, see [1].
In Section 5.1 we apply Equation (14) to Moody's historic default data for corporate bonds in order to estimate the asset/APP correlation for every rating class, assuming that the underlying corporate bond portfolios can be analytically approximated by a homogeneous reference portfolio; see the beginning of Section 3.
We will also estimate a *systematic APP* process; see Section 5.2.

5.1 Example (Part I): APP Correlations from Moody's Data

The following Table 1 shows the relative default frequency of corporate bonds according to the Moody's report [12] from 2002, including default data from 1970 to 2001.

Rating	1970	1971	1972	1973	1974	1975	1976	1977	1978	1979
Aaa	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
Aa	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
A	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
Baa	0,28%	0,00%	0,00%	0,47%	0,00%	0,00%	0,00%	0,28%	0,00%	0,00%
Ba	4,19%	0,43%	0,00%	0,00%	0,00%	1,04%	1,03%	0,53%	1,10%	0,49%
B	22,78%	3,85%	7,14%	3,77%	6,90%	5,97%	0,00%	3,28%	5,41%	0,00%
Caa	53,33%	13,33%	40,00%	44,44%	0,00%	0,00%	0,00%	50,00%	0,00%	0,00%
1980	**1981**	**1982**	**1983**	**1984**	**1985**	**1986**	**1987**	**1988**	**1989**	**1990**
0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,69%	0,00%
0,00%	0,00%	0,27%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
0,00%	0,00%	0,31%	0,00%	0,37%	0,00%	1,36%	0,00%	0,00%	0,61%	0,00%
0,00%	0,00%	2,78%	0,94%	0,87%	1,80%	1,78%	2,76%	1,26%	3,00%	3,37%
5,06%	4,49%	2,41%	6,31%	6,72%	8,22%	11,80%	6,27%	6,10%	9,29%	16,18%
33,33%	0,00%	27,27%	44,44%	100,00%	0,00%	23,53%	20,00%	28,57%	33,33%	53,33%
1991	**1992**	**1993**	**1994**	**1995**	**1996**	**1997**	**1998**	**1999**	**2000**	**2001**
0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,17%
0,29%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,12%	0,11%	0,39%	0,30%
5,43%	0,31%	0,57%	0,24%	0,70%	0,00%	0,19%	0,64%	1,03%	0,91%	1,19%
14,56%	9,05%	5,86%	3,96%	4,99%	1,49%	2,16%	4,15%	5,88%	5,42%	9,35%
36,84%	27,91%	30,00%	5,26%	12,07%	13,99%	14,67%	15,09%	20,05%	18,15%	32,50%

Table 1: Moody's corporate bond defaults [12].

The table shows observed default frequencies per rating class and year. Table 2 shows the result when applying the estimation procedure from Section 3.

Rating	Mean	Stand.Dev.	Impl.Ass.Corr.
Aaa	0,0005%	0,0031%	21,17%
Aa	0,0030%	0,0134%	20,17%
A	0,0194%	0,0585%	18,80%
Baa	0,1263%	0,2557%	17,23%
Ba	0,8228%	1,1180%	15,90%
B	5,3623%	4,8879%	16,41%
Caa	34,9453%	21,3696%	32,08%
Mean	**5,8971%**		**20,25%**

Rating	Mean	Stand.Dev.	Ass.Corr.Orig.
Aaa	0,0000%	0,0000%	not observed
Aa	0,0216%	0,1220%	31,50%
A	0,0138%	0,0556%	22,89%
Baa	0,1528%	0,2804%	15,95%
Ba	1,2056%	1,3277%	13,00%
B	6,5256%	4,6553%	11,77%
Caa	24,7322%	21,7857%	42,51%
Mean	**4,6645%**		**22,94%**

Table 2: First and second moments according to Table 1 and estimated asset correlations

Recall that for every rating class - according to (13) - the asset correlation ϱ is determined by equation (14) with m_p and s_p^2 being the mean value and variance according to the default history as given in Table 1. Hereby, the upper table reports on the result when smoothing the historic means and standard deviations by a linear regression on a logarithmic scale. The lower table shows the result of the same calculation but with the original sample moments. For Aaa-rated bonds no defaults have been observed, such that the lower table shows 'not observed' for the Aaa-asset correlation estimate. Our conclusion from the result of our calculations (Table 2) is as follows:

Given that our model assumptions are not taking us too far away from the 'real world', our calculations show that the Basel II level of 20% correlation is often close to the estimated correlation. However in more than half of the rating classes 20% correlation is conservative.

5.2 Example (Part II): Implied Systematic APP Process

One assumption which could at first sight seem to be critical, is the way we treated the underlying systematic APP process $Y_1, Y_2, Y_3, ...$; see Section 3. There, we assumed these variables to be independent. In a more realistic approach one would probably prefer to model these systematic variables by means of an *autoregressive process*, e.g., with time lag 1 (i.e. an AR(1)-process). However, in our model we are *not* thinking in terms of Y being a *macroeconomic* factor, for which an autoregressive modelling would be recommended. Our Y reflects the 'instantanous dependency' between borrower's ability to pay and does not refer to some time-lagged macroeconomic effect.

Moreover, we can get the process of realizations $Y_1, Y_2, Y_3, ...$ of the APP-factor Y back by a simple least-squares fit. For this purpose, we used an L^2-solver for calculating $y_1, ..., y_n$ with

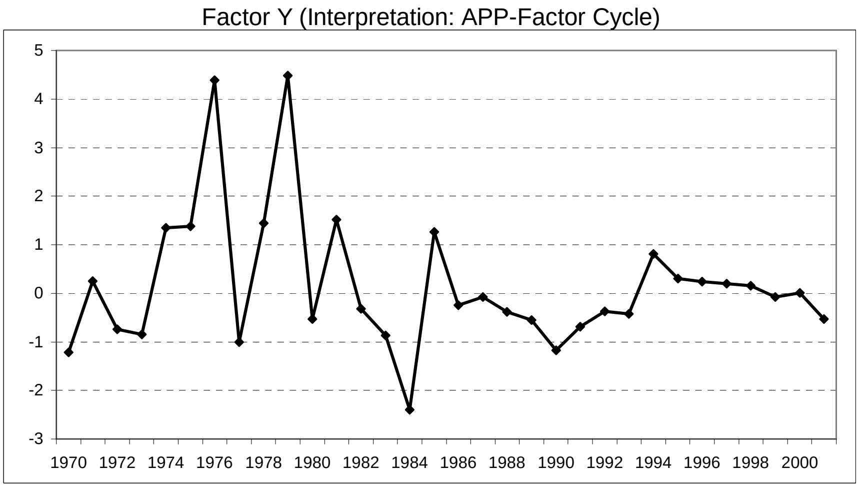
$$\sqrt{\sum_{j=1}^{32}\sum_{i=1}^{7}|p_{ij} - g_i(y_j)|^2} \;=\; \min_{(v_1,...,v_n)} \sqrt{\sum_{j=1}^{32}\sum_{i=1}^{7}|p_{ij} - g_i(v_j)|^2}\;,$$

where p_{ij} refers to the observed historic default frequency in rating class i in year j, according to Table 1, and $g_i(v_j)$ is defined by

$$g_i(v_j) \;=\; N\left[\frac{N^{-1}[\overline{p}_i] - \sqrt{\varrho_i}v_j}{\sqrt{1-\varrho_i}}\right] \qquad (i = 1, ..., 7;\; j = 1, ..., 32)\;,$$

reflecting Equation (10) where i denotes rating class i. Note that, ϱ_i refers to the just estimated asset correlations for the rating classes according to Table 2, lower table.
Figure 1 shows the resulting 'APP-factor cycle' and the time-dependent overall mean of the default frequencies in Moody's corporate bond universe.

The result is very intuitive:
Comparing the APP-factor cycle $y_1, ..., y_n$ with the historic mean default path, one can see that any systematic 'APP-downturn' corresponds to an increase of default frequencies.



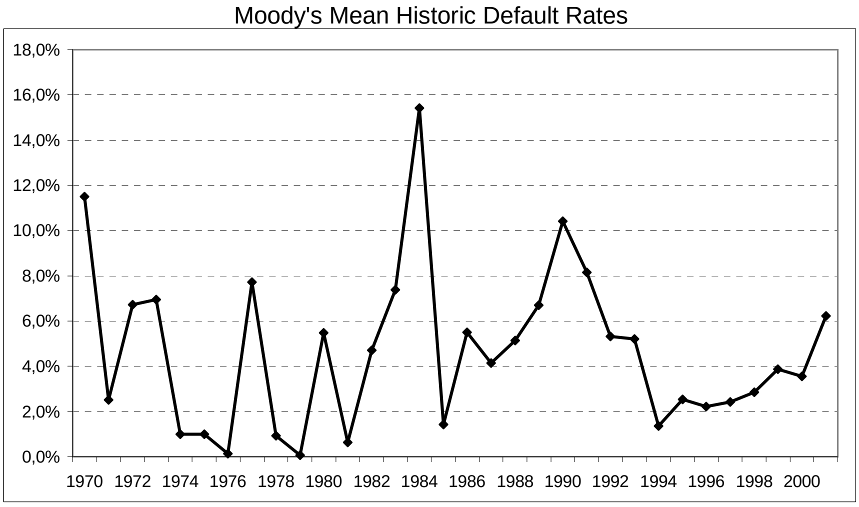


Figure 1: Systematic APP process and underlying mean default frequency path

5.3 Example (Part III): Correlation between Segments

Recalling our results from Section 4, we can consider every rating class as a segment and apply Equations (15) ('basic version') and (20) ('multi index approach') in order to estimate the segment correlation ϱ between rating classes.

5.3.1 Basic Version

Based on Equation (15), we can calculate the asset/APP correlation between rating classes Baa and Ba. From Table 2 we have

- $\overline{p}_{Baa} = 15{,}28$bps and $\overline{p}_{Ba} = 120{,}56$bps.

The empirical covariance can be obtained from the time series in Table 1:

- Cov$[(p_{Baa,j})_{j=1,\ldots,32}, (p_{Ba,j})_{j=1,\ldots,32}] = 0{,}00104\%$.

We then apply Equation (15) and obtain

- $\varrho = 5{,}60\%$.

This example indicates that the Basel II assumption of a one-factor model with a uniform asset correlation of 20% is violated as soon as we consider correlations between different segments.

5.3.2 Multi Index Approach

Using the same notation as in Section 4, the intra-segment correlation ϱ_k for segment k, where k ranges over all seven rating classes, is given in Table 2. In our example, we work with the lower table in Table 2, which is based on the original moments (without regression).
As an example, consider rating classes 4 (Baa) and 5 (Ba). From Table 2 we have

- $\overline{p}_{Baa} = 15{,}28$bps and $\overline{p}_{Ba} = 120{,}56$bps;
- $\varrho_{Baa} = 15{,}95\%$ and $\varrho_{Ba} = 13{,}00\%$.

For calculating ϱ, we first of all need to calculate the empirical covariance of the default frequency time series of rating classes Baa and Ba. In the previous section, the covariance of the time series of default rates in Table 1 has been estimated as $0{,}00104\%$
Dividing the covariance by the respective standard deviations yields a correlation between the two time series of about 28%. Next, we solve Equation (20) for ϱ and get

- $\varrho = 38{,}7\%$

as the correlation between the two factors. So much regarding an example calculation. Now let us interpret our result in terms of the 20%-correlation assumption of Basel II.
Following Basel II, a 'pure' one-factor approach is claimed to be sufficient for capturing *diversification effects*. Under this hypotheses, the correlation between systematic factors Y_k must be equal to $\varrho = 1$; cp. Equations (16) and (17). In contrast, our calculations above indicate that ϱ in fact is much lower than 100%. It is easily verified by means of analogous calculations, that this observation remains true even when dropping the multi-segment approach (allowing for different R^2's in different segments) by assuming ϱ_k to be constant for all segments k.

The assumption of a uniform asset correlation of 20% as made in the current draft of the new capital accord underestimates diversification benefits and does not provide any incentive to optimise the portfolio's risk profile by investing in different 'risk segments' like countries or industries.

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