Macro-Element Hierarchical Riesz Bases

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Abstract. We show that a nested sequence of C^r macro-element spline spaces on quasi-uniform triangulations gives rise to hierarchical Riesz bases of Sobolev spaces $H^s(\Omega)$, $1 < s < r+\frac{3}{2}$, and $H_0^s(\Omega)$, $1 < s < \sigma+\frac{3}{2}$, $s \notin \mathbb{Z}+\frac{1}{2}$, as soon as there is a nested sequence of Lagrange interpolation sets with uniformly local and bounded basis functions, and, in case of $H_0^s(\Omega)$, the nodal interpolation operators associated with the macro-element spaces are boundary conforming of order σ. In addition, we provide a brief review of the existing constructions of C^1 Largange type hierarchical bases.

Keywords: Hierarchical bases, Riesz bases, macro-elements, bivariate splines, Jackson inequality, Bernstein inequality

1 Introduction

Smooth macro-element spaces are among most practically useful spaces of piecewise polynomial splines in two and three space dimensions, see [20]. They are available on arbitrary polygonal domains and possess stable local bases and hence full approximation order. Some of them are refinable and therefore suitable for the multiresolution analysis [5–7, 9, 11–15, 17, 26, 27], with applications in particular to multilevel methods in numerical partial differential equations and surface modelling.

Given a sequence of nested spline spaces $S_0 \subset S_1 \subset \cdots \subset S_n \subset \cdots$, and corresponding nested interpolation sets $\Xi_0 \subset \Xi_1 \subset \cdots \subset \Xi_n \subset \cdots$ with Lagrange bases $\{B_\xi^{(n)}\}_{\xi\in\Xi_n}$, *hierarchical bases* are obtained from the appropriately re-scaled functions

$$B_\xi^{(n)}, \quad \xi \in \Xi_n \setminus \Xi_{n-1}, \qquad n = 0, 1, \ldots \ (\Xi_{-1} := \emptyset).$$

The most famous example is given by the piecewise linear basis functions (hat functions), where the hierarchical basis is used for the multilevel preconditioning of the discretised second order elliptic equations [31]. The effectiveness of this method is related to the Riesz basis (or "stability") property of this hierarchical basis in the Sobolev spaces $H^s(\Omega)$ and $H_0^s(\Omega)$, $1 < s < \frac{3}{2}$. For elliptic equations of forth order, stability in $H^2(\Omega)$ and $H_0^2(\Omega)$ is needed, and this can be achieved by C^1 hierarchical bases [12] that are Riesz bases in the range $1 < s < \frac{5}{2}$. In fact, as noted in [21], bases with stability in $H^s(\Omega)$ with as large as possible range of s is advantageous, in particular when an elliptic operator includes parts of different order. Moreover, a good preconditioning effect is expected when s corresponding to a given variational problem lies in the central part of the stability interval.

In this paper we study general conditions for the nested sequences of macro-element spline spaces to give rise to Riesz bases in $H^s(\Omega)$ and $H^s_0(\Omega)$. The main results (see Theorem 5) show that the stability range $1 < s < r + \frac{3}{2}$ in $H^s(\Omega)$ is guaranteed for refinable C^r macro-elements on quasi-uniform triangulations in $\mathbb{R}^2$ if the Lagrange bases $\{B^{(n)}_\xi\}_{\xi \in \Xi_n}$ are uniformly local and bounded, and the nodal bases of the macro-element spaces are also uniformly bounded. Moreover, the same stability range (up to the half-integer values) is obtained in $H^s_0(\Omega)$ if the macro-element nodal (Hermite) interpolation operators Π_n are *boundary conforming* of order r in the sense that for any function f vanishing on the boundary of Ω together with its derivatives up to order r, the interpolants $\Pi_n f$ have the same property.

The paper is organised as follows. In Section 2 we list some auxiliary results on K-functionals, interpolation spaces and Sobolev spaces $H^s(\Omega)$ and $H^s_0(\Omega)$. Section 3 is devoted to Bernstein and Jackson inequalities for bivariate splines, including the Bernstein inequality in $H^s(\Omega)$ for spline spaces possessing stable local bases, and error bounds for the macro-element nodal interpolation of functions in Sobolev spaces of integer order. General results on hierarchical bases of Lagrange type are given in Section 4, whereas C^1 macro-element spaces where such bases are known are reviewed in Section 5. In particular, we verify that the sequence of nested triangulations suggested in [12] is quasi-uniform.

Throughout we employ the usual notation $a \lesssim b$ and $a \sim b$ to indicate that the inequality (respectively, the double inequality) includes bounding constants which are not of interest. The parameters on which these constants may depend are either explicitly mentioned or clear from the context.

2 Preliminaries

We denote by $W^k_p(\Omega)$, $k \in \mathbb{N}$, $1 \le p \le \infty$, the usual Sobolev spaces on a bounded Lipschitz domain Ω. The space $C^k(\Omega) \subset W^k_\infty(\Omega)$ consists of all k times continuously differentiable functions f on the closure of Ω, with $\|f\|_{C^k(\Omega)} = \|f\|_{W^k_\infty(\Omega)}$. The space $W^k_2(\Omega)$ is also denoted by $H^k(\Omega)$, with $H^0(\Omega) := L_2(\Omega)$. It is a Hilbert space with inner product

$$\langle f, g \rangle_{H^k(\Omega)} = \langle f, g \rangle_{L_2(\Omega)} + \sum_{|\alpha|=k} \Big\langle \frac{\partial^\alpha f}{\partial x^\alpha}, \frac{\partial^\alpha g}{\partial x^\alpha} \Big\rangle_{L_2(\Omega)},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+$ is a multi-index, with $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

Let X and $Y \subset X$ be two Hilbert spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y = \|\cdot\|_X + |\cdot|_Y$, respectively, where $|\cdot|_Y$ is a seminorm. The K-functional is defined for each $f \in X$ and $t > 0$ by

$$K_{XY}(f, t) := \inf_{g \in Y} \|f - g\|_X + t|g|_Y,$$

or equivalently (see [22, Remark 4.8]) by the same expression with $|g|_Y$ replaced by $\|g\|_Y$.

One of the key properties of the K-functional is the following Jackson type inequality.

Lemma 1. *Let S be linear subspace of X. Suppose that for some $t > 0$,*

$$\inf_{s\in S} \|g - s\|_X \le t|g|_Y, \quad \text{for all} \quad g \in Y.$$

Then for any $f \in X$,

$$\inf_{s\in S} \|f - s\|_X \le K_{XY}(f,t).$$

Proof. Indeed,

$$\inf_{s\in S} \|f - s\|_X \le \inf_{g\in Y} \inf_{s\in S} (\|f - g\|_X + \|g - s\|_X) \le K_{XY}(f,t)$$

if the assumption holds. □

According to the K-method [1], the interpolation space $[X,Y]_\theta$, $0 < \theta < 1$, consists of all $f \in X$ for which the functional

$$|f|_{\theta;K} = \left(\int_0^\infty \left(t^{-\theta} K_{XY}(f,t)\right)^2 \frac{dt}{t} \right)^{1/2} \tag{1}$$

is finite. Given any $\alpha > 1$, by splitting the domain of integration $(0,\infty)$ into the intervals $(\alpha^{-n-1}, \alpha^{-n})$, $n = 0, 1, \ldots$, and $(1,\infty)$, and using standard properties of the K-functional, it is easy to show that

$$|f|_{\theta;K} \sim \left(\sum_{n=0}^\infty \left[\alpha^{n\theta} K_{XY}(f, \alpha^{-n})\right]^2 \right)^{1/2}, \tag{2}$$

where the constants of equivalence depend only on θ and α.

The *k-th modulus of smoothness* of $f \in L_p(\Omega)$, $0 < p \le \infty$, is defined by

$$\omega_k(f,t)_p = \sup_{|\delta|<t} \|\Delta_\delta^k f\|_{L_p(\Omega_{k\delta})},$$

where $|\delta|$ denotes the Euclidean length of $\delta \in \mathbb{R}^n$, $\Omega_{k\delta} := \{x \in \Omega : x + j\delta \in \Omega,\ j = 0, \ldots, k\}$, and

$$(\Delta_\delta^k f)(x) := \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} f(x + j\delta), \quad x \in \mathbb{R}^n,$$

is the usual difference operator. By [27, Theorem 1], the modulus of smoothness is equivalent to the K-functional,

$$\omega_k(f,t)_2 \sim K_{L_2,H^k}(f,t^k), \qquad t > 0. \tag{3}$$

Therefore, in view of Lemma 1, error bounds for functions in Sobolev spaces immediately lead to Jackson type estimates in terms of the modulus of smoothness.

The Sobolev spaces $H^s(\Omega)$ of a fractional order $s > 0$ can be defined as interpolation spaces

$$H^s(\Omega) = \left[L_2(\Omega),\ H^k(\Omega)\right]_\theta,$$

where $s = k\theta$, k integer, $0 < \theta < 1$. In view of (1) and (3),

$$|f|_{H^s(\Omega)} \sim \Big(\int_0^\infty (t^{-s}\omega_k(f,t)_2)^2 \,\frac{dt}{t}\Big)^{1/2}. \tag{4}$$

Let $C_c^\infty(\Omega)$ be the linear space of all infinitely differentiable functions on Ω with compact support contained in Ω. We use $H_0^s(\Omega)$ to denote the closure of $C_c^\infty(\Omega)$ in $H^s(\Omega)$. It is well known [19] that $C_c^\infty(\Omega)$ is dense in $H^s(\Omega)$ if and only if $s \le \frac{1}{2}$. If $s > \frac{1}{2}$ and the boundary of Ω is smooth, then $H_0^s(\Omega)$ is a proper subspace of $H^s(\Omega)$ given by

$$H_0^s(\Omega) = \Big\{u \in H^s(\Omega) : \frac{\partial^\alpha u}{\partial x^\alpha} = 0 \text{ on } \partial\Omega, \text{ for all } 0 \le |\alpha| < s - \frac{1}{2},\ \alpha \in \mathbb{Z}^n\Big\},$$

see [19, Theorem 11.5]. Hence, $H_0^s(\Omega) = H^s(\Omega)$ if $s \le \frac{1}{2}$ and $H_0^s(\Omega) = H^s(\Omega) \cap H_0^{s_0}(\Omega)$, where $s_0 = \lceil s - \frac{1}{2}\rceil$ if $s > \frac{1}{2}$. According to [19, Theorem 11.6] the spaces $H_0^s(\Omega)$ of fractional order $s \notin \mathbb{Z} + \frac{1}{2}$ can be obtained from the integer order spaces $H_0^k(\Omega)$, $k > s$, by interpolation

$$H_0^s(\Omega) = \left[L_2(\Omega), H_0^k(\Omega)\right]_\theta, \quad \theta = \frac{s}{k}, \quad s \notin \mathbb{Z} + \frac{1}{2}. \tag{5}$$

For $s \in \mathbb{Z}+\frac{1}{2}$ a description of the interpolation spaces $H_{00}^s(\Omega) := \left[L_2(\Omega), H_0^k(\Omega)\right]_\theta$, $\theta = \frac{s}{k}$, can be found in [19, Theorem 11.7].

For a domain $\Omega \subset \mathbb{R}^2$ with piecewise smooth boundary in the sense of [16, p. 34], which includes the case of Lipschitz polygonal domains, the interpolation property (5) has been shown in [32]. As shown in [16], $H_0^s(\Omega)$, $s \notin \mathbb{Z} + \frac{1}{2}$, in this case coincides with the space $\tilde{H}^s(\Omega)$ of all those functions $f \in H^s(\Omega)$ whose extension to $\mathbb{R}^2$ by zero belongs to $H^s(\mathbb{R}^2)$. See also [2] for (5) in the case of a bounded Lipschitz domain in any space dimensions and integer s.

3 Bernstein and Jackson inequalities for bivariate splines

We first recall standard definitions from the theory of bivariate piecewise polynomial splines, see [20] for more details.

Let Ω be polygonal domain in $\mathbb{R}^2$ and Δ a finite collection of (closed) triangles whose union coincides with Ω. We assume that the intersection of any two triangles in Δ is empty, or a common vertex, or a common edge of them. Then Δ is a *triangulation* of Ω. The length of an edge e of Δ is denoted by $|e|$. Let ξ be the set of all edges of Δ. The maximum length of the edges of Δ, denoted by $h = h_\Delta = \sup_{e\in\xi} |e|$, is called the *diameter* or *mesh size* of Δ. We denote the smallest angle of the triangles $T \in \Delta$ by β_Δ, and set

$$\gamma_\Delta = \min\{\operatorname{diam} T : T \in \Delta\}/h_\Delta.$$

A family of triangulations is called *regular* if $\beta_\Delta \geq \beta > 0$ for every Δ in the family. A regular family is said to be *quasi-uniform* if $\gamma_\Delta \geq \gamma > 0$ for every Δ.

For any positive integer d, let $S_d(\Delta)$ denote the space of all piecewise polynomials of degree d with respect to Δ. In other words, $s \in S_d(\Delta)$ if and only if, on each triangle $T \in \Delta$, s agrees with a polynomial in $\mathbb{P}_d$, the space of all bivariate polynomials of total degree at most d. For any $r = 0, 1, \ldots, d-1$, let

$$S_d^r(\Delta) := S_d(\Delta) \cap C^r(\Omega)$$

be the space of all piecewise polynomials of degree d and smoothness r with respect to Δ.

Let $\{s_1, \ldots, s_N\}$ be a basis for a linear space $S \subset S_d(\Delta)$. We say that the basis is *m-local* if for each $i = 1, \ldots, N$ there is a triangle $T_i \in \Delta$ such that $\operatorname{supp} s_i \subset \operatorname{star}^m(T_i)$. Here $\operatorname{star}^k(T) := \operatorname{star}(\operatorname{star}^{k-1}(T))$ for $k \geq 2$, where if U is the union of a cluster of triangles, then $\operatorname{star}(U) = \operatorname{star}^1(U)$ is the union of all triangles in Δ that have a non-empty intersection with U. A basis is called *local* if it is m-local for some m.

Suppose that $\{\lambda_1, \ldots, \lambda_N\} \subset S^*$ is the dual basis, that is,

$$\lambda_i s_j = \begin{cases} 1, & i = j, \\ 0, & \text{otherwise.} \end{cases}$$

A basis $\{s_1, \ldots, s_N\}$ for $S \subset S_d(\Delta)$ is said to be a *stable local basis* [8] if for an integer m and positive constants C_1, C_2,

(a) $\{s_1, \ldots, s_N\}$ is m-local,
(b) $|\lambda_i s| \leq C_1 \|s\|_{L_\infty(\operatorname{star}^m(T_i))}$ for all $s \in S$, $i = 1, \ldots, N$, and
(c) $\|s_i\|_{L_\infty(\Omega)} \leq C_2$, $i = 1, \ldots, N$.

Any stable local basis is L_p*-stable* for all $1 \leq p \leq \infty$ after appropriate renorming, that is, for any $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N$,

$$k_1 C_2^{-1} \|\alpha\|_{l_p} \leq \Big\| \sum_{i=1}^N \alpha_i \frac{s_i}{|\operatorname{supp} s_i|^{1/p}} \Big\|_{L_p(\Omega)} \leq k_2 C_1 \|\alpha\|_{l_p}, \quad 1 \leq p \leq \infty,$$

where k_1, k_2 are some constants depending only on p, r, d and m, and $|M|$ denotes the area of a set $M \subset \mathbb{R}^2$.

3.1 Bernstein inequality

Functions in subspaces of $S_d^r(\Delta)$ possessing a stable local basis satisfy a Bernstein type inequality in the norm of $H^s(\Omega)$ for all $0 < s < r + \frac{3}{2}$.

Theorem 1 (Bernstein Inequality). *Suppose that $S \subset S_d^r(\Delta)$ has a stable local basis $\{\phi_i\}_{i\in I}$. Then for any $f \in S$,*

$$\|f\|_{H^s(\Omega)} \lesssim h_\Delta^{-s}\|f\|_{L_2(\Omega)}, \qquad 0 < s < r + \frac{3}{2}, \tag{6}$$

where the bounding constant depends only on $s, r, d, \beta_\Delta, \gamma_\Delta$ and the parameters m, C_1, C_2 of the stable local basis.

Under slightly different assumptions on S, a proof of the Bernstein inequality can be found in [27], see also [29]. We provide a proof based on the following lemma.

Lemma 2 ([17, Lemma 2.2]). *Let $f \in S_d^r(\Delta)$. Then $f \in H^s(\Omega)$ for all $s < r + \frac{3}{2}$, and*

$$\|f\|_{H^s(\Omega)} \lesssim h_\Delta^{-s}\|f\|_{L_2(\Omega)}, \qquad 0 < s < r + \frac{3}{2}, \tag{7}$$

where the bounding constant depends only on $s, r, d, \beta_\Delta, \gamma_\Delta$ and the number of triangles $T \in \Delta$ in the support of f.

Proof (of Theorem 1). Since $\{\phi_i\}_{i\in I}$ is a stable local basis, the functions $\psi_i = |\operatorname{supp}\phi_i|^{-1/2}\phi_i$, $i \in I$, form an L_2-stable basis for S. In particular $\|\psi_i\|_{L_2(\Omega)} \le M$, where M depends only on the parameters m, C_1, C_2 of the stable local basis.

Let $f = \sum_{i\in I} c_i\psi_i$, so that $\|f\|^2_{L_2(\Omega)} \sim \sum_{i\in I} |c_i|^2$. Choose an integer $k > s$. Since the basis $\{\psi_i\}_{i\in I}$ is m-local, it is not difficult to see that

$$I_k(f,\delta)^2 \lesssim \sum_{i\in I} |c_i|^2 I_k(\psi_i,\delta)^2, \quad \text{where } I_k(f,\delta) := \|\Delta_\delta^k f\|_{L_2(\Omega_{k\delta})}, \quad \delta \in \mathbb{R}^2.$$

Hence,

$$\omega_k(f,t)_2^2 \lesssim \sum_{i\in I} |c_i|^2 \omega_k(\psi_i,t)_2^2,$$

and by (4),

$$|f|^2_{H^s(\Omega)} \lesssim \sum_{i\in I} |c_i|^2 \int_0^\infty (t^{-s}\omega_k(\psi_i,t)_2)^2 \,\frac{dt}{t} \sim \sum_{i\in I} |c_i|^2 |\psi_i|^2_{H^s(\Omega)}.$$

By applying Bernstein inequality (7) to the locally supported functions ψ_i and using the L_2-stability of the basis $\{\psi_i\}_{i\in I}$, in particular, uniform L_2-boundedness of ψ_i, we obtain

$$\|f\|^2_{H^s(\Omega)} \lesssim h_\Delta^{-2s} \sum_{i\in I} |c_i|^2 \|\psi_i\|^2_{L_2(\Omega)} \lesssim h_\Delta^{-2s} \sum_{i\in I} |c_i|^2 \lesssim h_\Delta^{-2s}\|f\|^2_{L_2(\Omega)},$$

which completes the proof. □

3.2 Jackson inequality for macro-element spline spaces

We restrict our attention to the macro-element spaces, see [20, Section 5.10], because of the availability of boundary conforming interpolation operators that allow appropriate treatment of subspaces with zero boundary conditions.

Recall that a linear functional λ is called a *nodal functional* provided λf is a scalar multiple of the value of f or its (directional) partial derivative at some point $\eta = \eta(\lambda) \in \mathbb{R}^2$, that is $\lambda f = \gamma \frac{\partial^{\nu+\mu} f}{\partial\sigma^\nu \partial\tau^\mu}(\eta)$, for suitable $\nu, \mu \in \mathbb{Z}_+$, $\eta \in \Omega$, unit vectors σ, τ, and a scaling coefficient $\gamma \in \mathbb{R}$. The number $\kappa(\lambda) = \nu + \mu$ is called the *order* of λ.

A collection $\mathcal{N} = \{\lambda_i\}_{i=1}^N$ is called a *nodal determining set* for a spline space $S \subset S_d(\Delta)$ if every $s \in S$ is $\kappa(\lambda)$ times continuously differentiable at $\eta(\lambda)$, and $\lambda s = 0$ for all $\lambda \in \mathcal{N}$ implies $s \equiv 0$. $\mathcal{N}$ is called a *nodal minimal determining set* (NMDS) for S if there is no smaller nodal determining set. In other words, $\mathcal{N}$ is an NMDS if it is a basis for the dual S^* of S. Let $\{s_i\}_{i=1}^N$ be the basis of S dual to $\mathcal{N}$, called the *nodal basis*.

We will work with spaces of splines that are defined on triangulations $\Delta_R = \bigcup_{K\in\Delta} K_R$ obtained from a given partition Δ of Ω into polygonal cells K by applying some refinement process to each $K \in \Delta$. Examples are provided by Clough-Tocher and Powell-Sabin splits of the triangles of a triangulation Δ of Ω. We assume that each K is star-shaped with respect to a disk. We denote by χ_K the *chunkiness parameter* $\operatorname{diam} K/\rho_{\max}$ of K, where $\rho_{\max}$ is the maximum radius of disks with respect to which K is star-shaped [3, Section 4.3]. Recall that χ_K is bounded in terms of the minimum angle of K if K is a triangle. We set $\chi_\Delta := \max_{K\in\Delta} \chi_K$.

For each cell $K \in \Delta$, we define

$$\mathcal{N}_K = \{\lambda \in \mathcal{N} : \eta(\lambda) \in K\}.$$

We call $S \subset S_d(\Delta_R)$ a *macro-element space* provided there is a NMDS $\mathcal{N}$ for S such that for each $K \in \Delta$, $S|_K$ is uniquely determined from the values $\{\lambda s\}_{\lambda\in\mathcal{N}_K}$. It is easy to see that the support of a basis function s_i in a macro-element space is contained in the union of all $K \in \Delta$ containing $\eta(\lambda_i)$. For each $\lambda_i \in \mathcal{N}$, we choose the scaling coefficient γ to be equal to $\gamma_i = \operatorname{diam}(T_i)^{\kappa(\lambda_i)}$, where $T_i \in \Delta_R$ is a triangle containing $\eta(\lambda_i)$. Note that $\operatorname{diam}(T_i) \sim \operatorname{diam}(T')$ for any other triangle $T' \in \Delta_R$ sharing a vertex with T_i, with the constant of equivalence depending only on β_{Δ_R}, see [20, Section 4.7], and $\operatorname{diam}(T_i) \sim \operatorname{diam}(K)$, where $T_i \subset K \in \Delta$, and the constant of equivalence depends only on β_{Δ_R} and $\nu_{\Delta_R} := \max_{K\in\Delta} |K_R|$. Then by Markov inequality [20, Theorem 2.32] $|\lambda_i s| \le C_1 \|s\|_{L_\infty(T_i)}$ for any $s \in S$, where C_1 depends only on d, $\kappa(S) := \max_i \kappa(\lambda_i)$ and β_{Δ_R}. It follows that $\{s_i\}_{i=1}^N$ is a stable local basis for S with parameters depending only on d, $\kappa(S)$, β_{Δ_R} and ν_{Δ_R} as soon as $\|s_i\|_{L_\infty(\Omega)} \le C_2$, $i = 1, \dots, N$, for some constant C_2.

The interpolation operator $\Pi : C^{\kappa(S)}(\Omega) \to S$ is defined by

$$\Pi f = \sum_{i=1}^N \lambda_i(f) s_i. \tag{8}$$

By the duality of the basis functions s_i, it is clear that $\Pi s = s$ for all $s \in S$. In particular, Π reproduces polynomials of degree at most k if $\mathbb{P}_k \subset S$. The definition of the macro-element space implies that the *local interpolation operators* $\Pi_K : C^{\kappa(S)}(K) \to S|_K$,

$$\Pi_K f = \sum_{i:\, \eta(\lambda_i) \in K} \lambda_i(f) s_i$$

satisfy $\Pi_K f = (\Pi f)|_K$ for $f \in C^{\kappa(S)}(\Omega)$.

We say that the interpolation operator Π is *boundary conforming* of order σ if the homogeneous boundary conditions of order σ are preserved by the interpolant, that is, if

$$\frac{\partial^{\nu+\mu} f}{\partial x^\nu \partial y^\mu} = 0 \quad \text{on } \partial\Omega, \qquad \text{for all} \quad \nu, \mu \geq 0, \quad \nu + \mu \leq \sigma,$$

implies

$$\Pi f \in S_{0,\sigma} := \{s \in S : \frac{\partial^{\nu+\mu} s}{\partial x^\nu \partial y^\mu} = 0 \text{ on } \partial\Omega, \text{ for all } \nu, \mu \geq 0,\ \nu + \mu \leq \sigma\}.$$

The proof of the following version of the Jackson inequality follows the scheme used in [3, Section 4.4], where it is proved for *finite elements*, thus making an assumption of *affine equivalence* of the spaces $S|_K$, $K \in \Delta$. In place of affine equivalence, we only assume that the nodal basis is uniformly bounded, see (9).

Theorem 2 (Jackson Inequality). *Let $S \subset S_d^r(\Delta_R)$ be a macro-element space such that $\mathbb{P}_k \subset S$ for some $1 \leq k \leq d$, and $\kappa(S) \leq k - 1$. Assume that its nodal basis $\{s_i\}_{i=1}^N$ satisfies*

$$\|s_i\|_{L_\infty(\Omega)} \leq C_2, \qquad i = 1, \ldots, N. \tag{9}$$

Then for every $f \in H^{k+1}(\Omega)$,

$$\|f - \Pi f\|_{H^\nu(\Omega)} \leq C h_\Delta^{k+1-\nu} |f|_{H^{k+1}(\Omega)}, \quad \nu = 0, \ldots, \min\{r, k\} + 1, \tag{10}$$

where C depends only on d, β_{Δ_R}, ν_{Δ_R}, χ_Δ and C_2.

Proof. Recall that by Sobolev embedding theorem any function $f \in H^{k+1}(\Omega)$ belongs (after possible modification on a set of zero measure) to $C^{k-1}(\Omega)$. This implies that Πf is well defined for all $f \in H^{k+1}(\Omega)$, and $f - \Pi f \in H^{r+1}(\Omega)$ since $S_d^r(\Delta_R) \subset H^{r+1}(\Omega)$.

Given any $K \in \Delta$, we define

$$\hat{K} := \left\{ \frac{x}{\operatorname{diam}(K)} : x \in K \right\}.$$

Then $\operatorname{diam} \hat{K} = 1$ and hence $|\hat{K}| \leq \pi/4$. For any function g defined on K we set $\hat{g}(y) := g(\operatorname{diam}(K) y)$, $y \in \hat{K}$. The functions

$$\hat{s}_i := \widehat{s_i|_K}, \qquad \text{for all } i \text{ such that } \lambda_i \in \mathcal{N}_K,$$

form a basis for the spline space $\hat{S}_K := \{\hat{s} : s \in S|_K\}$ on $\hat{K}$, with its dual basis given by the linear functionals $\hat{\lambda}_i(\hat{g}) := \lambda_i(g)$, $g \in C^{k-1}(K)$. Since $\operatorname{diam}(T_i) \sim \operatorname{diam}(K)$, we have

$$\hat{\lambda}_i \hat{g} = \operatorname{diam}(T_i)^{\nu+\mu} \frac{\partial^{\nu+\mu} g}{\partial\sigma^\nu \partial\tau^\mu}(\eta) \sim \frac{\partial^{\nu+\mu} \hat{g}}{\partial\sigma^\nu \partial\tau^\mu}(\operatorname{diam}(K)^{-1}\eta),$$

and it follows that

$$|\hat{\lambda}_i(g)| \le \hat{C}_1 \|g\|_{C^{k-1}(\hat{K})}, \quad g \in C^{k-1}(\hat{K}), \quad \lambda_i \in \mathcal{N}_K, \tag{11}$$

where $\hat{C}_1$ depends only on β_{Δ_R}, ν_{Δ_R} and d. Note that by Sobolev inequality [3, Section 4.3],

$$\|g\|_{C^{k-1}(\hat{K})} \lesssim \|g\|_{H^{k+1}(\hat{K})}, \quad g \in H^{k+1}(\hat{K}) \subset C^{k-1}(\hat{K}), \tag{12}$$

where the bounding constant depends only on k and the chunkiness parameter $\chi_{\hat{K}}$ $(= \chi_K)$.

We define the interpolation operator $\mathit{\Pi}_{\hat{K}} : C^{k-1}(\hat{K}) \to \hat{S}_K$ by

$$\mathit{\Pi}_{\hat{K}} g := \sum_{i:\, \lambda_i \in \mathcal{N}_K} \hat{\lambda}_i(g) \hat{s}_i.$$

By (9) we get

$$\|\hat{s}_i\|_{L_2(\hat{K})} \le \frac{\sqrt{\pi}}{2} \|\hat{s}_i\|_{L_\infty(\hat{K})} \le \frac{\sqrt{\pi}}{2} C_2,$$

which in view of the Bernstein inequality (7) leads to

$$\|\hat{s}_i\|_{H^{r+1}(\hat{K})} \le \hat{C}_2, \tag{13}$$

where $\hat{C}_2$ depends only on d, r, β_{Δ_R}, $|K_R|$ and C_2.

The inequalities (11) and (13) imply that the operator $\mathit{\Pi}_{\hat{K}} : C^{k-1}(\hat{K}) \to H^{r+1}(\hat{K})$ is uniformly bounded, i.e.,

$$\|\mathit{\Pi}_{\hat{K}} g\|_{H^{r+1}(\hat{K})} \le \hat{C}_3 \|g\|_{C^{k-1}(\hat{K})}, \tag{14}$$

where the constant $\hat{C}_3$ depends only on $\hat{C}_1$, $\hat{C}_2$, d and $|K_R|$. Indeed, let $g \in C^{k-1}(\hat{K})$. Then $\mathit{\Pi}_{\hat{K}} g \in \hat{S}_K \subset W_\infty^{r+1}(\hat{K}) \subset H^{r+1}(\hat{K})$. Clearly, $|\mathcal{N}_K|$ does not exceed a constant C' depending only on d and $|K_R|$. In view of (11) and (13),

$$\|\mathit{\Pi}_{\hat{K}} g\|_{H^{r+1}(\hat{K})} \le \sum_{i:\, \lambda_i \in \mathcal{N}_K} |\hat{\lambda}_i(g)|\ \|\hat{s}_i\|_{H^{r+1}(\hat{K})} \le C' \hat{C}_1 \hat{C}_2 \|g\|_{C^{k-1}(\hat{K})}.$$

We now show that for every $K \in \mathit{\Delta}$ and $g \in H^{k+1}(K)$,

$$|g - \mathit{\Pi}_K g|_{H^\nu(K)} \lesssim \operatorname{diam}(K)^{k+1-\nu} |g|_{H^{k+1}(K)}, \quad 0 \le \nu \le \min\{r, k\} + 1, \tag{15}$$

where the constant in the bound depends only on d, β_{Δ_R}, ν_{Δ_R}, χ_K and C_2. If $g \in H^{k+1}(K)$, then $\hat{g} \in H^{k+1}(\hat{K})$ and, by the Bramble-Hilbert lemma [3, Section 4.3] there exists a polynomial $p \in \mathbb{P}_k$ such that

$$\|\hat{g} - p\|_{H^\ell(\hat{K})} \lesssim |\hat{g}|_{H^{k+1}(\hat{K})}, \qquad 0 \le \ell \le k+1, \tag{16}$$

where the bounding constant depends only on k and the chunkiness parameter $\chi_{\hat{K}}$ $(= \chi_K)$. Let $m = \min\{r, k\}$. Since $\Pi_{\hat{K}} p = p$, we have by (14), (12) and (16),

$$\begin{aligned}\|\hat{g} - \Pi_{\hat{K}}\hat{g}\|_{H^{m+1}(\hat{K})} &\le \|\hat{g} - p\|_{H^{m+1}(\hat{K})} + \|\Pi_{\hat{K}}(p - \hat{g})\|_{H^{m+1}(\hat{K})} \\ &\lesssim \|\hat{g} - p\|_{H^{k+1}(\hat{K})} + \|p - \hat{g}\|_{C^{k-1}(\hat{K})} \\ &\lesssim \|\hat{g} - p\|_{H^{k+1}(\hat{K})} + \|p - \hat{g}\|_{H^{k+1}(\hat{K})} \\ &\lesssim |\hat{g}|_{H^{k+1}(\hat{K})},\end{aligned}$$

and (15) follows since

$$|g - \Pi_K g|_{H^\nu(K)} = \mathrm{diam}(K)^{1-\nu} |\hat{g} - \Pi_{\hat{K}}\hat{g}|_{H^\nu(\hat{K})},$$

$|\hat{g} - \Pi_{\hat{K}}\hat{g}|_{H^\nu(\hat{K})} \le \|\hat{g} - \Pi_{\hat{K}}\hat{g}\|_{H^{m+1}(\hat{K})}$, and

$$|\hat{g}|_{H^{k+1}(\hat{K})} = \mathrm{diam}(K)^k |g|_{H^{k+1}(K)}.$$

The estimate (10) follows from (15) because

$$\|f - \Pi f\|^2_{H^\nu(\Omega)} = \sum_{K \in \Omega} \sum_{i=0}^{\nu} |f|_K - \Pi_K f|_K|^2_{H^i(K)}, \ |f|^2_{H^{k+1}(\Omega)} = \sum_{K \in \Omega} |f|_K|^2_{H^{k+1}(K)}$$

and $h_\Delta = \max_{K \in \Delta} \mathrm{diam}(K)$. □

It follows from the proof that the local estimate

$$\|f - \Pi_K f\|_{H^\nu(K)} \le C \,\mathrm{diam}(K)^{k+1-\nu} |f|_{H^{k+1}(K)}, \quad K \in \Delta,$$

holds for all $\nu = 0, \ldots, k+1$ as soon as $f|_K \in H^{k+1}(K)$, without the restriction $\nu \le r+1$.

Note that the estimate

$$\inf_{g \in S} \|f - g\|_{H^\nu(\Omega)} \le C h_\Delta^{k+1-\nu} |f|_{H^{k+1}(\Omega)}, \quad f \in H^{k+1}(\Omega),$$

can be obtained by using quasi-interpolation operators for any spline spaces S with a stable local basis, see [20] or [10]. Even though Theorem 2 is only applicable to macro-element spaces, its importance for the results below about Riesz bases in $H_0^s(\Omega)$ is that it leads to the estimate

$$\inf_{g \in S_{0,\sigma}} \|f - g\|_{H^\nu(\Omega)} \le C h_\Delta^{k+1-\nu} |f|_{H^{k+1}(\Omega)}, \quad f \in H_0^{k+1}(\Omega), \tag{17}$$

as soon as the interpolation operator Π is boundary conforming of some order $\sigma \le r$, which is normally the case for the macro-elements.

Corollary 1. *In addition to the assumptions of Theorem 2 suppose that the interpolation operator Π is boundary conforming of order $\sigma \le r$. Then the estimate* (17) *holds for all $\nu = 0, \ldots, \min\{r, k\} + 1$, where C depends only on d, β_{Δ_R}, ν_{Δ_R}, χ_Δ and C_2.*

4 General theory of hierarchical Riesz bases

Recall that a basis $\{\phi_n\}_{n=1}^{\infty}$ for a Hilbert space H is said to be a *Riesz basis* if for any $c \in \ell_2$,

$$\Big\| \sum_{n=1}^{\infty} c_n \phi_n \Big\|_H \sim \Big(\sum_{n=1}^{\infty} c_n^2 \Big)^{1/2}.$$

Suppose that S_n, $n = 0, 1, 2 \ldots$, is a *nested sequence* of finite dimensional subspaces of a Hilbert space H, that is

$$S_0 \subset S_1 \subset \ldots \subset S_n \subset \ldots \quad n = 0, 1, 2, \ldots. \tag{18}$$

We assume that $\cup_{n=0}^{\infty} S_n$ is dense in H and set $S_{-1} := \{0\}$. Then every element $f \in H$ can be represented as a convergent series $\sum_{n=0}^{\infty} f_n$ in H with $f_n \in S_n$. For $n = 0, 1, 2, \ldots$, let P_n be a linear projection from S_n onto S_{n-1}, and let W_n be the complement space, that is, $P_n(W_n) = \{0\}$ and $S_n = S_{n-1} + W_n$. In particular, $W_0 = S_0$.

We will use the following general result about construction of Riesz bases for certain subspaces of H using stable bases of W_n. More standard statements of this type are usually restricted to the case when the projectors P_n are uniformly bounded, see e.g. [6].

Theorem 3 ([18]). *Assume that for some $v > 0$ and $\rho > 1$,*

$$\|P_{n+1} \cdots P_m f\|_H \lesssim \rho^{v(m-n)} \|f\|_H, \qquad f \in S_m, \tag{19}$$

for all $m, n = 0, 1, 2, \ldots$ with $n < m$. Let $s > v$ and let H_s be a linear subspace of H which itself is a Hilbert space with norm $\|\cdot\|_{H_s}$ satisfying

$$\big\|f\big\|_{H_s} \sim \inf_{f_n \in S_n:\, f = \sum_{n=0}^{\infty} f_n} \Big(\sum_{n=0}^{\infty} \big[\rho^{ns} \|f_n\|_H\big]^2 \Big)^{1/2}, \qquad f \in H_s. \tag{20}$$

Suppose that for each $n = 0, 1 \ldots$, $W_n \subset H_s$ and there is a stable basis $\{\phi_k^{(n)}\}_{k \in K_n}$ for W_n in the sense that

$$\Big\| \sum_{k \in K_n} c_k \phi_k^{(n)} \Big\|_H \sim \Big(\sum_{k \in K_n} c_k^2 \Big)^{1/2}, \tag{21}$$

with constants of equivalence independent of n. Then $\bigcup_{n=0}^{\infty} \{\rho^{-ns} \phi_k^{(n)}\}_{k \in K_n}$ is a Riesz basis for H_s.

Assumption (20) of Theorem 3 can often be verified with the help of the following theorem. Although it can be derived from more general results in e.g. [4, 22] (see also [28]), we provide here a short and self-contained proof based on arguments similar to those in [27, Theorem 6] and [7, Corollary 5.2].

Theorem 4. *Let H and $H' \subset H$ be Hilbert spaces with norms $\|\cdot\|_H$ and $\|\cdot\|_{H'} = \|\cdot\|_H + |\cdot|_{H'}$, where $|\cdot|_{H'}$ is a seminorm. Suppose that for some $\alpha > 1$ and $0 < \lambda < 1$ nested finite dimensional linear subspaces $S_n \subset H$ satisfy the Jackson inequality*

$$\inf_{s\in S_n} \|f - s\|_H \lesssim \alpha^{-n}|f|_{H'}, \quad f \in H', \qquad n = 0, 1, \dots, \tag{22}$$

and the Bernstein inequality in the norm $\|\cdot\|_{\lambda;K}$ of the interpolation space $[H, H']_\lambda$,

$$\|s\|_{\lambda;K} \lesssim \alpha^{n\lambda}\|s\|_H, \qquad s \in S_n. \tag{23}$$

Then for any $0 < \theta < \lambda$,

$$\|f\|_{\theta;K} \sim \inf_{f_n\in S_n:\, f=\sum_{n=0}^\infty f_n} \Big(\sum_{n=0}^\infty \big[\alpha^{n\theta}\|f_n\|_H\big]^2\Big)^{1/2}, \quad f \in [H, H']_\theta, \tag{24}$$

where the constants of equivalence depend only on α, the difference $\lambda - \theta$ and the bounding constants in (22) and (23).

Proof. Recall from (2) that

$$\|f\|_{\theta;K} \sim \|f\| := \|f\|_H + \Big(\sum_{n=0}^\infty \big[\alpha^{n\theta}K_{H,H'}(f, \alpha^{-n})\big]^2\Big)^{1/2}.$$

We will show that $\|f\| \sim \|f\|^*$, where $\|f\|^*$ denotes the right hand side of (24).

We first prove that $\|f\|^* \lesssim \|f\|$. Let $f \in H$. It follows from (22) by Lemma 1 that there exists a sequence of elements $f_n \in S_n$ such that

$$\|f - f_n\|_H \lesssim K_{H,H'}(f, \alpha^{-n}), \qquad n = 0, 1, \dots.$$

Then

$$\|f_n - f_{n-1}\|_H \le \|f_n - f\|_H + \|f_{n-1} - f\|_H \lesssim K_{H,H'}(f, \alpha^{-n}), \quad n \ge 1,$$

and $\|f_0\|_H \lesssim \|f\|_H + K_{H,H'}(f, 1)$. If $\|f\| < \infty$, then $\|f - f_n\|_H \to 0$ when $n \to \infty$ and hence

$$f = \sum_{n=0}^\infty (f_n - f_{n-1}), \qquad f_{-1} = 0,$$

where $f_n - f_{n-1} \in S_n$ since $S_{n-1} \subset S_n$, which implies

$$\|f\|^* \le \Big(\sum_{n=0}^\infty \big[\alpha^{n\theta}\|f_n - f_{n-1}\|_H\big]^2\Big)^{1/2} \lesssim \|f\|.$$

We now proceed to showing the opposite inequality $\|f\| \lesssim \|f\|^*$. Let $f = \sum_{n=0}^\infty f_n$ with some $f_n \in S_n$. By (23) we have for $t \in [\alpha^{-(j+1)}, \alpha^{-j}]$,

$$K_{H,H'}(f_n, t)^2 \le K_{H,H'}(f_n, \alpha^{-j})^2 \lesssim \alpha^{-2\lambda j}|f_n|^2_{\lambda,K} \lesssim (t\alpha^n)^{2\lambda}\|f_n\|^2_H. \tag{25}$$

Let $0 < \theta < \lambda$. Then

$$\sum_{j=0}^{\infty} \alpha^{2\theta j} K_{H,H'}(f, \alpha^{-j})^2 \leq 2(A+B),$$

where

$$A = \sum_{j=0}^{\infty} \alpha^{2j\theta} \left(\sum_{n=0}^{j} K_{H,H'}(f_n, \alpha^{-j}) \right)^2, \quad B = \sum_{j=0}^{\infty} \alpha^{2j\theta} \left(\sum_{n=j+1}^{\infty} K_{H,H'}(f_n, \alpha^{-j}) \right)^2.$$

By (25) and Cauchy-Schwarz inequality,

$$\begin{aligned} A &\lesssim \sum_{j=0}^{\infty} \alpha^{2j\theta} \left(\sum_{n=0}^{j} \alpha^{(n-j)\lambda} \|f_n\|_H \right)^2 = \sum_{j=0}^{\infty} \alpha^{2j(\theta-\lambda)} \left(\sum_{n=0}^{j} \alpha^{n(\lambda-\theta)} \alpha^{n\theta} \|f_n\|_H \right)^2 \\ &\leq \sum_{j=0}^{\infty} \alpha^{2j(\theta-\lambda)} \sum_{n=0}^{j} \alpha^{n(\lambda-\theta)} \sum_{n=0}^{j} \alpha^{n(\lambda-\theta)} \alpha^{2n\theta} \|f_n\|_H^2. \end{aligned}$$

Since

$$\sum_{n=0}^{j} \alpha^{n(\lambda-\theta)} = \frac{\alpha^{(j+1)(\lambda-\theta)} - 1}{\alpha^{(\lambda-\theta)} - 1} \leq \frac{\alpha^{(\lambda-\theta)}}{\alpha^{(\lambda-\theta)} - 1} \cdot \alpha^{j(\lambda-\theta)},$$

we get

$$A \lesssim \sum_{j=0}^{\infty} \alpha^{-j(\lambda-\theta)} \sum_{n=0}^{j} \alpha^{n(\lambda-\theta)} \alpha^{2n\theta} \|f_n\|_H^2 = C_1 \sum_{n=0}^{\infty} \alpha^{2n\theta} \|f_n\|_H^2,$$

where $C_1 = \sum_{k=0}^{\infty} \alpha^{-k(\lambda-\theta)} = \frac{1}{1-\alpha^{-(\lambda-\theta)}}$.

The bound $K_{H,H'}(f_n, \alpha^{-j}) \leq \|f_n\|_H$ and the Cauchy-Schwarz inequality imply

$$\begin{aligned} B &\leq \sum_{j=0}^{\infty} \alpha^{2j\theta} \left(\sum_{n=j+1}^{\infty} \|f_n\|_H \right)^2 = \sum_{j=0}^{\infty} \alpha^{2j\theta} \left(\sum_{n=j+1}^{\infty} \alpha^{\frac{-n\theta}{2}} \alpha^{\frac{-n\theta}{2}} \alpha^{n\theta} \|f_n\|_H \right)^2 \\ &\leq \sum_{j=0}^{\infty} \alpha^{2j\theta} \sum_{n=j+1}^{\infty} \alpha^{-n\theta} \sum_{n=j+1}^{\infty} \alpha^{-n\theta} \alpha^{2n\theta} \|f_n\|_H^2 \\ &= \frac{\alpha^{-\theta}}{1-\alpha^{-\theta}} \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \alpha^{(j-n)\theta} \alpha^{2n\theta} \|f_n\|_H^2 \leq C_2 \sum_{n=0}^{\infty} \alpha^{2n\theta} \|f_n\|_H^2, \end{aligned}$$

where $C_2 = \frac{\alpha^{-\theta-1}}{(1-\alpha^{-\theta})(1-\alpha^{-1})}$. Combining the above estimates for A and B yields $\|f\| \lesssim \|f\|^*$. □

We will use Theorems 3 and 4 with $H = L_2(\Omega)$ and $H_s = H^s(\Omega)$ or $H_0^s(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is an arbitrary polygonal domain, and $\{S_n\}_{n=0}^{\infty}$ is a nested sequence of macro-element spline spaces.

A sequence of triangulations $\{\Delta_n\}_{n=0}^\infty$ of Ω is said to be *nested* if each Δ_{n+1} is a refinement of Δ_n, that is Δ_{n+1} is obtained from Δ_n by subdividing the triangles of Δ_n. Then obviously $S_d^r(\Delta_{n+1}) \subset S_d^r(\Delta_n)$, so that $\{S_d^r(\Delta_n)\}_{n=0}^\infty$ is a nested sequence of spaces. However, certain subspaces $S_n \subset S_d^r(\Delta_n)$ may also be nested, see for example [9, 11, 13].

Recall that a sequence of triangulations $\{\Delta_n\}_{n=0}^\infty$ of Ω is *regular* if the minimum angle of all Δ_n remains bounded below by a positive constant $\beta > 0$ independent of n, and the triangulations Δ_n are quasi-uniform in the sense that there exist constants $\rho > 1$ and $c_1, c_2 > 0$ independent of n such that

$$c_1\rho^{-n} \le \operatorname{diam} T \le c_2\rho^{-n}, \qquad T \in \Delta_n. \tag{26}$$

Parameter ρ will be called the *refinement factor* of $\{\Delta_n\}_{n=0}^\infty$.

Recall that a finite set $\Xi \subset \Omega$ is said to be a *Lagrange interpolation set* for a finite dimensional linear space S of functions on Ω if $\#\Xi = \dim S$ and for each $\xi \in \Xi$ there is a unique function $B_\xi \in S$ satisfying $B_\xi(\eta) = \delta_{\xi,\eta}$ for all $\xi, \eta \in \Xi$, where $\delta_{\xi,\eta} = 1$ if $\xi = \eta$ and $\delta_{\xi,\eta} = 0$ otherwise. The set $\{B_\xi\}_{\xi\in\Xi}$ is a basis for S called the *Lagrange basis*.

A sequence of Lagrange interpolation sets $\{\Xi_n\}_{n=0}^\infty$ for the corresponding spaces S_n is said to be *nested* if

$$\Xi_0 \subset \Xi_1 \subset \ldots \subset \Xi_n \subset \ldots. \tag{27}$$

We are ready to formulate the main result of the paper.

Theorem 5. *Let $\{S_n\}_{n=0}^\infty$ be a nested sequence of spaces $S_n \subset S_d^r(\Delta_n)$, $r \ge 0$, with respect to a regular nested sequence of triangulations $\{\Delta_n\}_{n=0}^\infty$ of a polygonal domain $\Omega \subset \mathbb{R}^2$, with refinement factor $\rho > 1$, and let $\{\Xi_n\}_{n=0}^\infty$ be a nested sequence of Lagrange interpolation sets for the spaces S_n, with the corresponding Lagrange basis $\{B_\xi^{(n)}\}_{\xi\in\Xi_n}$ for S_n. Assume that the bases $\{B_\xi^{(n)}\}_{\xi\in\Xi_n}$ are uniformly local and bounded, that is they are m-local and satisfy $\|B_\xi^{(n)}\|_{L_\infty(\Omega)} \le M$, $\xi \in \Xi_n$, for some m, M independent of n.*

(a) Assume that the spaces S_n satisfy the Jackson inequality

$$\inf_{g\in S_n} \|f - g\|_{L_2(\Omega)} \lesssim \rho^{-n(k+1)}|f|_{H^{k+1}(\Omega)}, \quad f \in H^{k+1}(\Omega), \tag{28}$$

For some $k \in \mathbb{N}$ with $r < k \le d$. Then for any $s \in (1, r + \frac{3}{2})$ the set

$$\mathcal{B}_s := \bigcup_{n=0}^{\infty} \{\rho^{n(1-s)}B_\xi^{(n)}\}_{\xi\in\Xi_n\setminus\Xi_{n-1}}$$

is a Riesz basis for $H^s(\Omega)$.

(b) Moreover, if the spaces S_n, $n = 0, 1, \ldots$, satisfy the homogeneous boundary conditions of order $\sigma \le r$, that is

$$\frac{\partial^{\nu+\mu} g}{\partial x^\nu \partial y^\mu} = 0 \text{ on } \partial\Omega, \text{ for all } \nu, \mu \ge 0,\ \nu + \mu \le \sigma, \quad g \in S_n,$$

and (28) *holds for all* $f \in H_0^{k+1}(\Omega)$ *rather than for all* $f \in H^{k+1}(\Omega)$*, then* $\mathcal{B}_s$ *is a Riesz basis for* $H_0^s(\Omega)$ *if* $s \in (1, \sigma + \frac{3}{2}) \setminus (\mathbb{Z} + \frac{1}{2})$.

Proof. Under the assumptions of the theorem, the bases $\{B_\xi^{(n)}\}_{\xi \in \Xi_n}$ are stable and local in the sense of the definition in Section 3. Since $\operatorname{diam}(T) \sim \rho^{-n}$, $T \in \Delta_n$, the bases $\{\rho^n B_\xi^{(n)}\}_{\xi \in \Xi_n}$ are L_2-stable, which implies

$$\Big\| \sum_{\xi \in \Xi_n} c_\xi B_\xi^{(n)} \Big\|_{L_2(\Omega)} \sim \rho^{-n} \Big(\sum_{\xi \in \Xi_n} c_\xi^2 \Big)^{1/2}, \tag{29}$$

for any real numbers c_ξ, with constants of equivalence independent of n.

Let $0 < s < r + \frac{3}{2}$. We choose a number $\bar{s}$ such that $s < \bar{s} < r + \frac{3}{2}$. By Theorem 1, since the spaces S_n possess stable local bases, we obtain the Bernstein inequality

$$\|g\|_{H^{\bar{s}}(\Omega)} \lesssim \rho^{n\bar{s}} \|g\|_{L_2(\Omega)}, \qquad g \in S_n.$$

By Theorem 4, applied with $\alpha = \rho^{k+1}$, $\lambda = \bar{s}/(k+1) < 1$ and $\theta = s/(k+1)$, we see that under the assumptions of part (a) condition (20) of Theorem 3 is satisfied for $H = L_2(\Omega)$, $H' = H^{k+1}(\Omega)$ and $H_s = H^s(\Omega) = [L_2(\Omega), H^{k+1}(\Omega)]_\theta$. Similarly, under the assumptions of part (b) condition (20) follows from Theorem 4 with $H = L_2(\Omega)$, $H' = H_0^{k+1}(\Omega)$ and $H_s = [L_2(\Omega), H_0^{k+1}(\Omega)]_\theta$.

We now verify the other assumptions of Theorem 3. The density of $\cup_{n=0}^\infty S_n$ in $H = L_2(\Omega)$ follows from the Jackson inequality (28) since both $H^{k+1}(\Omega)$ and $H_0^{k+1}(\Omega)$ are dense in $L_2(\Omega)$. Furthermore, let $I_n : C(\Omega) \to S_n$, $n = 0, 1, \ldots$, be the Lagrange interpolation operator

$$I_n f := \sum_{\xi \in \Xi_n} f(\xi) B_\xi^{(n)}.$$

We set $P_n := I_{n-1}|_{S_n}$, $n \geq 1$, and $P_0 := 0$. Then $P_n : S_n \to S_{n-1}$ is a linear projection, and, in view of the nestedness (27) of $\{\Xi_n\}_{n=0}^\infty$, we have $P_{n+1} \cdots P_m = I_n|_{S_m}$ for all $m > n$. Let $g \in S_m$ and $h := P_{n+1} \cdots P_m g$. Then $g = \sum_{\xi \in \Xi_m} g(\xi) B_\xi^{(m)}$ and $h = \sum_{\xi \in \Xi_n} g(\xi) B_\xi^{(n)}$. By (29) and (27) we obtain

$$\begin{aligned} \|h\|_{L_2(\Omega)}^2 &\lesssim \rho^{-2n} \sum_{\xi \in \Xi_n} |g(\xi)|^2 \leq \rho^{-2n} \sum_{\xi \in \Xi_m} |g(\xi)|^2 \\ &\lesssim \rho^{2(m-n)} \|g\|_{L_2(\Omega)}^2, \end{aligned}$$

which implies (19) with $H = L_2(\Omega)$ and $v = 1$. Because of (27) the sets

$$\{\rho^n B_\xi^{(n)} : \xi \in \Xi_n \setminus \Xi_{n-1}\}, \quad n = 0, 1, \ldots \quad (\Xi_{-1} = 0)$$

form L_2-stable bases for the complement spaces W_n. Since $W_n \subset S_n \subset H^s(\Omega)$ for all $s < r + \frac{3}{2}$ by Theorem 1, an application of Theorem 3 with $v = 1$ completes the proof of part (a). Under the assumptions of part (b) it is easy to see that

$S_n \subset \tilde{H}^s(\Omega) = H_0^s(\Omega)$ for all $s < \sigma + \frac{3}{2}$, $s \notin \mathbb{Z} + \frac{1}{2}$, and Theorem 3 implies that $\mathcal{B}_s$ is a Riesz basis for $[L_2(\Omega), H_0^{k+1}(\Omega)]_{s/(k+1)}$ for all $1 < s < \sigma + \frac{3}{2}$. The statement of part (b) follows in view of the description (5) of these interpolation spaces in Section 2. □

Note that in the case $r = \sigma = 0$ the condition (20) of Theorem 3 for $H = L_2(\Omega)$ and $H_s = H_0^s(\Omega)$, $s < \frac{3}{2}$, can be verified with the help of [25, Corollary 3] without using interpolation spaces.

The argumentation of Theorem 5 for $\Omega \subset \mathbb{R}^d$ would lead to the Riesz basis for $H^s(\Omega)$ with the expectable range $\frac{d}{2} < s < r + \frac{3}{2}$. Indeed, (29) then holds with $\rho^{-\frac{dn}{2}}$ replacing ρ^{-n}, and hence Theorem 3 is applicable with $v = \frac{d}{2}$.

The standard C^0 piecewise linear hierarchical basis [31] is, after appropriate scaling, a Riesz basis of $H^s(\Omega)$ $s \in (1, \frac{3}{2})$ in two dimensions, see [21]. Clearly, Theorem 5 applies to this case, where the triangulations Δ_n are obtained by the uniform refinement of an initial triangulation of Ω, $\rho = 2$, S_n is either $S_1^0(\Delta_n)$ (for $H^s(\Omega)$) or its subspace $\{s \in S_n : s|_{\partial\Omega} = 0\}$ (for $H_0^s(\Omega)$), and Ξ_n is either the set of all vertices of Δ_n or the set of all interior vertices, respectively. The Jackson inequality (28) for $k = 1$ follows from Theorem 2 since $S_1^0(\Delta_n)$ are macro-element spaces with uniformly bounded basis functions, $\mathbb{P}_1 \subset S_1^0(\Delta_n)$, and the interpolation operator Π is boundary confirming of order $\sigma = 0$.

In the next section we provide a brief review of the existing constructions of C^1 Lagrange type hierarchical Riesz bases for Sobolev spaces $H^s(\Omega)$, $s \in (1, \frac{5}{2})$, and $H_0^s(\Omega)$, $s \in (1, \frac{3}{2}) \cup (\frac{3}{2}, \frac{5}{2})$. Note that C^1 hierarchical bases of Hermite type are also known [5, 26]. They form Riesz bases for $H^s(\Omega)$, $s \in (2, \frac{5}{2})$.

5 C^1 Lagrange hierarchical Riesz bases for Sobolev spaces

Spline spaces $S_n \subset S_d^r(\Delta_n)$ and Lagrange interpolation sets Ξ_n satisfying the hypotheses of Theorem 5 give rise to hierarchical Riesz bases for $H^s(\Omega)$, $s \in (1, r + \frac{3}{2})$, respectively $H_0^s(\Omega)$, $s \in (1, \sigma + \frac{3}{2}) \setminus (\mathbb{Z} + \frac{1}{2})$. However, specific constructions are only available for $r = 0, 1$. In this section we review such constructions of the spaces S_n in the case $r = 1$. We do not describe the corresponding sets Ξ_n as they are quite technical, and the interested reader is instead referred to the original literature.

5.1 Piecewise cubics on triangulated quadrangulations

The first construction of C^1 Lagrange hierarchical bases has been suggested in [12], where the nested spline spaces are the macro element spaces of C^1 piecewise cubic polynomials on the triangulations (see [20, Section 6.5]) obtained by adding two diagonals to the quadrilaterals of a *checkerboard* quadrangulation of any polygonal domain, which means that all interior vertices of the quadrangulation are of degree 4 and quadrilaterals can be coloured black and white in such a way that any two quadrilaterals sharing an edge have opposite colours. The corresponding nodal basis satisfies (9) with a constant C_2 dependent only on the minimum angle of the triangles $T \in \Delta_R$ and the interpolation operator Π is boundary conforming of order 1.

Nested spaces are obtained by the *triadic refinement* of the quadrilaterals and their subtriangles illustrated in Figures 1 and 2. More precisely, Let $Q = \langle v_1, v_2, v_3, v_3 \rangle$ be a quadrilateral and let $p_1 = 1/3(2v_1 + v_2)$, $p_2 = 1/3(v_1 + 2v_2)$, $p_3 = 1/3(2v_2 + v_3)$, $p_4 = 1/3(v_2 + 2v_3)$, $p_5 = 1/3(2v_3 + v_4)$, $p_6 = 1/3(v_3 + 2v_4)$, $p_7 = 1/3(2v_4 + v_1)$, $p_8 = 1/3(v_4 + 2v_1)$, $p_9 = 1/3(v_1 + 2\bar{v})$, $p_{10} = 1/3(v_2 + 2\bar{v})$, $p_{11} = 1/3(v_3 + 2\bar{v})$, $p_{12} = 1/3(v_4 + 2\bar{v})$, where $\bar{v}$ is the point of intersection of the diagonals of Q. The refinement is obtained by connecting the points p_1 and p_8 to p_9, p_2 and p_3 to p_{10}, p_4 and p_5 to p_{11}, p_6 and p_7 to p_{12}, and finally connecting the points $p_9, p_{10}, p_{11}, p_{12}$ together, as shown in Figure 1. Each of the 9 quadrilaterals is subdivided into 4 triangles by its diagonals as in Figure 2.

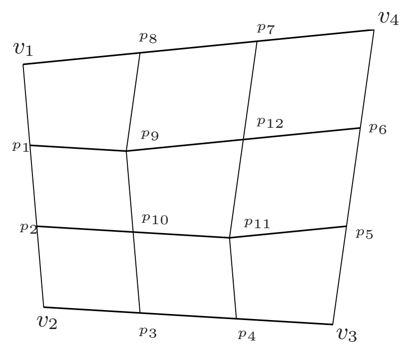

Fig. 1: A triadic refinement $\diamondsuit_Q$ of a quadrilateral Q.

Given an initial quadrangulation $\diamondsuit_0$ of Ω, this method generates a sequence of successively refined quadrangulations $\diamondsuit_0, \diamondsuit_1, \ldots, \diamondsuit_n, \ldots$, and triangulations $\Delta_0, \Delta_1, \ldots, \Delta_n, \ldots$, and the nested macro-element spaces are $S_n = S_3^1(\Delta_n)$. While the nestedness of the sequence of triangulations $\{\Delta_n\}_{n=0}^{\infty}$ is obvious, its regularity, which has not been fully addressed in [12], follows from Proposition 1 below. For the nested sequence of Lagrange interpolation sets $\{\Xi_n\}_{n=0}^{\infty}$ described in [12] all assumptions of Theorem 5 (b) are satisfied, with $r = \sigma = 1$, $k = 3$ and $\rho = 3$, which leads to a Riesz basis for $H_0^s(\Omega)$, $s \in (1, \frac{3}{2}) \cup (\frac{3}{2}, \frac{5}{2})$.

Proposition 1. *Each triangle $T \in \Delta_n$, $n \geq 2$, is similar to a triangle in Δ_1 with the scaling factor $\frac{1}{3^{n-1}}$.*

Proof. Consider the quadrangulation $\diamondsuit_Q$ of a quadrilateral Q obtained by the triadic refinement. It is easy to see that the quadrilateral $\langle p_9, p_{10}, p_{11}, p_{12} \rangle$ is similar to the parent quadrilateral $Q = \langle v_1, v_2, v_3, v_4 \rangle$, whereas $\langle p_1, p_2, p_{10}, p_9 \rangle$ is a parallelogram with side length $\frac{1}{3}$ of the size of the parent edge $\langle v_1, v_2 \rangle$, see Figure 1. Three other children of Q in similar position are also parallelograms.

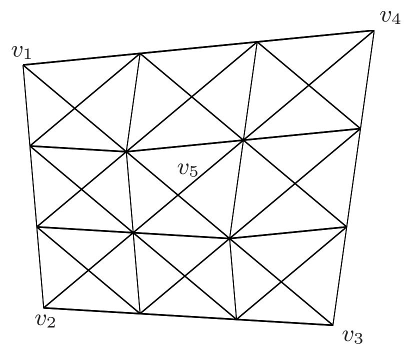

Fig. 2: The triangulation $\varDelta_Q$ of $\Diamond_Q$.

Let $\varDelta_Q$ be the triangulation of $\Diamond_Q$ shown in Figure 2. We observe that there are 8 different types of similar triangles in $\varDelta_Q$ as shown in Figure 3. The triangles of types $1, 2, 3$ and 4 are similar to their parent triangles (obtained from Q by splitting along its diagonals) with the coefficient $\frac{1}{3}$. The triangles of types $5, 6, 7$ and 8 will be referred to as "median" triangles because each of them has a side parallel to the median of its parent triangle and of length $\frac{2}{3}$ of that median, as illustrated in Figure 4, where the section $\langle v_1, v_2, v_5 \rangle$ of the triangulations $\varDelta_Q$ of Figure 2 is shown separately.

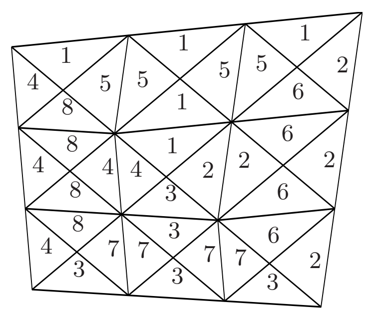

Fig. 3: Eight types of similar triangles in $\varDelta_Q$.

We now apply the next refinement step and look at the median subtriangle $\langle a, b, c \rangle$ of the median triangle in $\varDelta_Q$ as shown in Figure 5. We note that the dotted line $\langle q_1, q_2 \rangle$ is of length $\frac{2}{3}$ of the side $\langle p_1, p_9 \rangle$ of the parent which is parallel to the median $\langle p_1, p_2 \rangle$ of the grandparent. Hence the median of the median triangle $\langle a, b, c\rangle$ is of length $\frac{1}{4} \times \frac{2}{3} \times \frac{2}{3} = \frac{1}{9}$ of the median $\langle m, v_5\rangle$ of the grandparent $\langle v_1, v_2, v_5\rangle$. Therefore, the median subtriangle $\langle a, b, c\rangle$ is similar to the grandparent $\langle v_1, v_2, v_5\rangle$ with coefficient $\frac{1}{9}$.

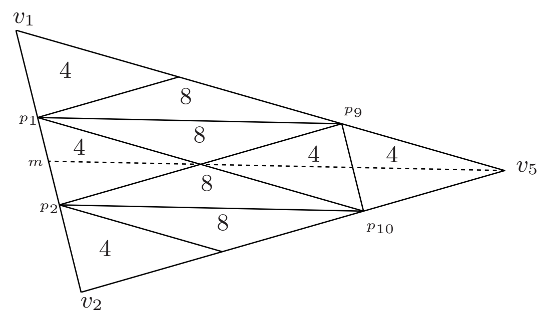

Fig. 4: The triangle $\langle v_1, v_2, v_5\rangle$, its median $\langle m, v_5\rangle$ and 9 children.

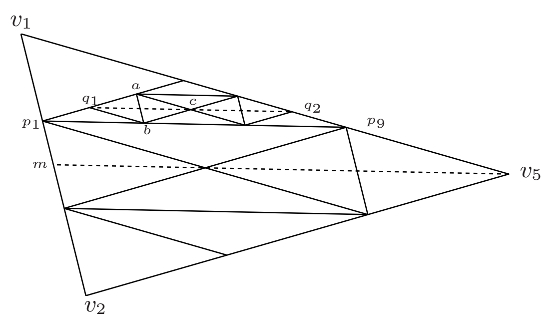

Fig. 5: The triangle $\langle a, b, c\rangle$ is similar to the parent $\langle v_1, v_2, v_5\rangle$ with coefficients $\frac{1}{9}$.

Let $T \in \Delta_n$, with $n \geq 2$. By applying the above observations recursively, we have two following cases: 1) T is similar to an ancestor $\tilde{T} \in \Delta_1$ with coefficient $\frac{1}{3^{n-1}}$. 2) T is similar to an ancestor $\hat{T} \in \Delta_0$ with coefficient $\frac{1}{3^n}$. But $\hat{T}$ has a child $\tilde{T} \in \Delta_1$ which is similar to $\hat{T}$ with coefficient $\frac{1}{3}$ and this implies that T is similar to $\tilde{T}$ with coefficient $\frac{1}{3^{n-1}}$. □

5.2 Piecewise quadratics on Powell-Sabin-6 splits

C^1 piecewise quadratic hierarchical bases are considered in [23]. Here, an initial checkerboard quadrangulation of Ω is first turned into a triangulation by adding one diagonal of each quadrilateral, and then each triangle is subdivided using a Powell-Sabin-6 (PS-6) split. To obtain a nested sequence of triangulations $\{\Delta_n\}_{n=0}^{\infty}$, a triadic refinement of the PS-6 split [30] is performed, see Figure 6. The nested spline spaces S_n are the C^1 piecewise quadratic Powell-Sabin macro-elements [20, Section 6.3]. Lagrange interpolation sets Ξ_n with the required properties are selected using a scheme which can be seen as a specific realisation of the interpolation method described in [24]. It is shown in [23] that this construction leads to a Riesz basis for $H^s(\Omega)$, $1 < s < \frac{5}{2}$, under the assumption that the triangulation sequence $\{\Delta_n\}_{n=0}^{\infty}$ is regular. Indeed, in this case Theorem 5 is applicable with $r = \sigma = 1$ and $k = 2$. We note however that this assumption does not seem easy to verify unless Δ_0 is a uniform triangulation, in which case $\rho = 3$. It is an open question whether an arbitrary polygonal domain Ω admits an initial triangulation such that the sequence of triangulations obtained by the triadic refinement of its PS-6 split is regular.

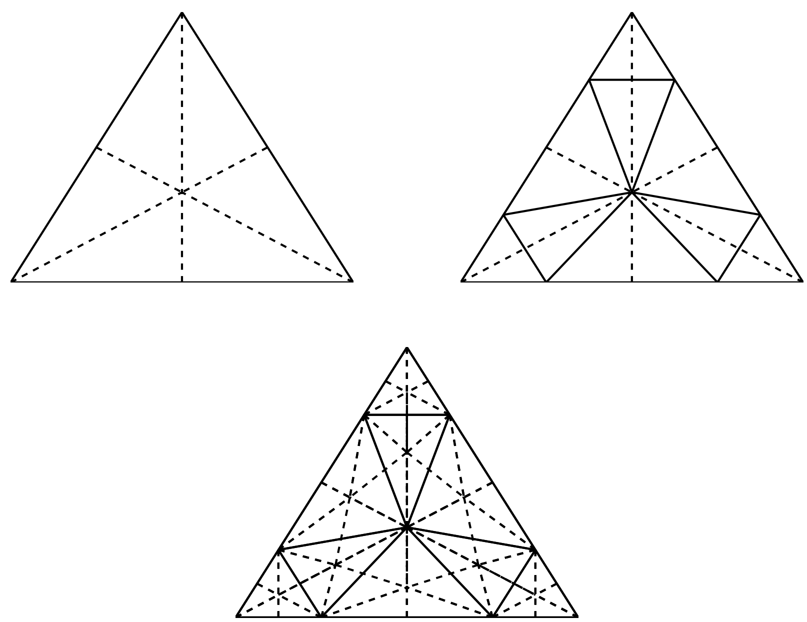

Fig. 6: The triadic refinement of the PS-6 split: A new vertex is placed at the position of the interior point in the PS-6 split and two new vertices on each edge.

5.3 Piecewise quadratics on mixed PS-6/PS-12 splits

In our paper [14] we construct C^1 piecewise quadratic hierarchical bases on arbitrary polygonal domains using nested sequences of triangulations and spline spaces introduced in [17]. Beginning with an arbitrary triangulation Δ_0 of Ω, a nested sequence of triangulations $\{\Delta_n\}_{n=0}^{\infty}$ is obtained by the standard uniform refinement, where the middle points of edges are connected to each other. An edge of Δ_n is said to be regular if it is shared by two triangles that form a parallelogram. Clearly, all boundary edges are irregular, but an interior edge may only be irregular if it overlaps a part of an edge of Δ_0. Furthermore, let Δ_n^* be the triangulation obtained by subdividing each triangle $T \in \Delta_n$ using the Powell-Sabin-6 split if all edges of T are regular, or the Powell-Sabin-12 split [20, Section 6.4] otherwise. For both PS-6 and PS-12 splits the central vertex is chosen at the barycentre of the triangle and the edge splitting vertices are at the midpoints of the edges. Then $\{\Delta_n^*\}_{n=0}^{\infty}$ is also a nested sequence of triangulations, as illustrated in Figure 7. It is obviously regular, with refinement factor $\rho = 2$.

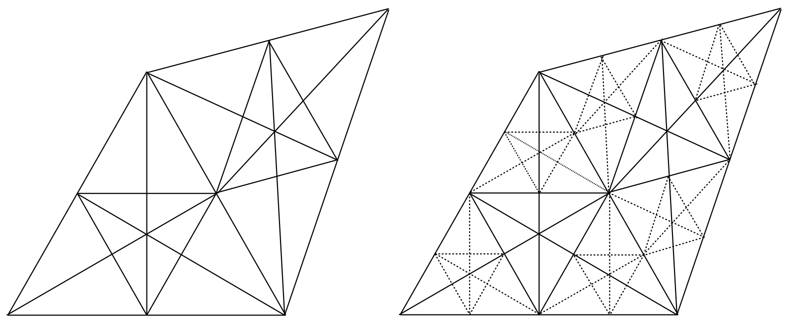

Fig. 7: An example to illustrate that Δ_{n+1}^* is a refinement of Δ_n^*.

The spline spaces are defined by

$$S_n = \left\{ s \in S_2^1(\Delta_n^*) : \frac{\partial s}{\partial e^{\perp}}\Big|_e \text{ is linear for each irregular edge } e \text{ of } \Delta_n \right\},$$

where $\frac{\partial s}{\partial e^{\perp}}$ denotes the normal derivative of s on e. It is easy to see that $\{S_n\}_{n=0}^{\infty}$ are nested macro-element spaces, their interpolation operators are boundary conforming of order 1, and $\mathbb{P}_2 \subset S_n$, $n = 0, 1, \ldots$. Let $P_n : S_n \to S_{n-1}$ be the orthogonal projector with respect to the inner product defined by

$$(f, g) = \sum_{e \in \mathcal{E}_n} (f, g)_e,$$

where $\mathcal{E}_n$ is the set of all edges of Δ_n and, for $e = \langle v_1, v_2 \rangle$,

$$\begin{aligned}(f,g)_e := \frac{1}{2^{2n}} \Big[& f(v_1)g(v_1) + \Big(f(v_1) + \frac{1}{4}\frac{\partial f}{\partial e}(v_1)\Big)\Big(g(v_1) + \frac{1}{4}\frac{\partial g}{\partial e}(v_1)\Big) \\ & + f(v_2)g(v_2) + \Big(f(v_2) - \frac{1}{4}\frac{\partial f}{\partial e}(v_2)\Big)\Big(g(v_2) - \frac{1}{4}\frac{\partial g}{\partial e}(v_2)\Big)\Big].\end{aligned}$$

It is shown in [17] that the projectors P_n satisfy (19) with

$$v = \log_2\left(\frac{2(1+\sqrt{13})}{3}\right) \approx 1.618,$$

and thus lead to a construction of Riesz bases in $H^s(\Omega)$ for $v < s < \frac{5}{2}$.

In [14] we present a construction of nested Lagrange interpolation sets for S_n and their subspaces with homogeneous boundary conditions of order 1, which leads to a Riesz basis for $H^s(\Omega)$, $s \in (1, \frac{5}{2})$ and $H_0^s(\Omega)$, $s \in (1, \frac{3}{2}) \cup (\frac{3}{2}, \frac{5}{2})$, by applying Theorem 5 with $r = \sigma = 1$, $k = 2$ and $\rho = 2$.

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