Refinable $C^2$ Piecewise Quintic Polynomials on Powell-Sabin-12 Triangulations

Oleg Davydov∗ Wee Ping Yeo†

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Abstract

We present a construction of nested spaces of $C^2$ macro-elements of degree 5 on triangulations of a polygonal domain obtained by uniform refinements of an initial triangulation and a Powell-Sabin-12 split.

1 Introduction

A sequence of piecewise polynomial spaces $S_0, S_1, \ldots, S_n, \ldots$ with respect to triangulations $\Delta_0, \Delta_1, \ldots, \Delta_n, \ldots$ of a domain $\Omega \subset \mathbb{R}^2$ is said to be nested if $S_n \subset S_{n+1}$ for all $n = 0, 1, \ldots$. Nested spaces of smooth piecewise polynomials (splines) are used in multilevel algorithms for surface compression [7, 9], nonlinear approximation [2, 4] and preconditioning of spline based finite element system matrices [1, 6, 10].

If the triangulations $\Delta_n$ are obtained by successive refinements of a starting triangulation $\Delta_0$, then the spaces $S_n = S^r_d(\Delta_n)$ of all $C^r$ splines of degree at most $d$ are nested. However, these spaces are known to possess stable local bases important for application only if $d$ is relatively large, $d \geq 3r + 2$, see [5]. Therefore much attention is paid to the macro-element spaces [8, Chapter 6] whose degree can be kept much lower at the expense of requiring that $\Delta_n$ is obtained from a general triangulation by splitting each triangle into subtriangles by various methods such as Clough-Tocher split, Powell-Sabin-6 or Powell-Sabin-12 split. Some $C^1$ macro-elements, such as piecewise quadratic Powell-Sabin-12 element or cubic Fraeijs de Veubeke-Sanders element, are refinable in the sense that nested spline spaces with stable local bases can be constructed with their help. However, refinable macro-elements of higher smoothness have not been known.

In this paper we propose the first construction of refinable $C^2$ macro-elements, whose degree 5 is substantially lower than degree 8 of the splines of [5] and degree 9 of the refinable $C^2$ spline spaces with stable dimension suggested in [3]. On a single macro-triangle our spaces coincide with the $C^2$ quintic macro-element of [11], although we

∗Department of Mathematics, University of Strathclyde, 26 Richmond Street, Glasgow G1 1XH, Scotland, oleg.davydov@strath.ac.uk
†Department of Mathematics, University of Strathclyde, 26 Richmond Street, Glasgow G1 1XH, Scotland, weeping.yeo@strath.ac.uk
obtain a simpler description of it (important for nestedness) in the case when the central point of the Powell-Sabin-12 split is placed at the barycentre of the macro-triangle. The nestedness of the spaces is achieved as in [3] by relaxing the $C^3$ smoothness conditions at the vertices of macro-triangles, which allows to break the ‘super-smoothness disks’ at the vertices into half-disks. The proposed macro-elements are easy to implement in the framework of the Bernstein-Bézier techniques because we provide explicit formulas for all $B$-coefficients which are not computed directly by the standard smoothness conditions.

The paper is organised as follows. In Section 2 we recall the basics of the Bernstein-Bézier techniques used throughout the paper. Sections 3 and 4 are devoted to the construction of nested spaces and stable minimal determining sets (leading to a stable local basis), as well as the proofs of the main results, whereas Section 5 provides a nodal minimal determining set and error bounds for the corresponding Hermite interpolation operator.

2 Bernstein-Bézier techniques

We recall basic notions of the Bernstein-Bézier techniques, see [8] for details. Given a triangle $T := \langle v_1, v_2, v_3 \rangle$, any bivariate polynomial $p$ of total degree $d$ can be uniquely represented in the form
\begin{equation}
 p = \sum_{i+j+k=d} c_{ijk} B_{ijk}^T,
\end{equation}
where $B_{ijk}^T$ are the Bernstein basis polynomials of degree $d$ associated with $T$. We refer to the representation (1) as the $B$-form of $p$ related to $T$. The $c_{ijk}$’s are called the $B$-coefficients of $p$, and the associated set of domain points is defined by
\begin{equation}
 D_{d,T} := \{ \xi_{ijk} := \frac{iv_1 + jv_2 + kv_3}{d} : i+j+k=d \}.
\end{equation}

Given a regular triangulation $\Delta = \{ T_i \}_{i=1}^N$ of a bounded connected polygonal domain $\Omega \subseteq \mathbb{R}^2$ and a positive integer $d$, we define the corresponding set of domain points by
\begin{equation}
 D_{d,\Delta} := \bigcup_{T \in \Delta} D_{d,T}.
\end{equation}
(Recall that a regular triangulation is such that the union of all triangles of $\Delta$ is $\bar{\Omega}$ and the intersection of any pair of triangles of $\Delta$ either consists of a common edge or a common vertex of both triangles or is empty.)

Let $S_0^d(\Delta)$ be the space of continuous splines of degree $d$ on $\Delta$,
\begin{equation}
 S_0^d(\Delta) := \{ s \in C^0(\Omega) : s|_{T_i} \in \mathbb{P}_d, \ i = 1, \ldots, N \},
\end{equation}
where $\mathbb{P}_d$ denotes the space of all bivariate polynomials of total degree at most $d$. Given $s \in S_0^d(\Delta)$ and $T \in \Delta$, there exists a unique set of coefficients $\{ c_{\xi} \}_{\xi \in D_{d,T}}$ such that
\begin{equation}
 s|_T = \sum_{\xi \in D_{d,T}} c_{\xi} B_{\xi}^{T,d},
\end{equation}
and each spline in $S^0_d(\Delta)$ is uniquely determined by its set of B-coefficients \( \{c_\xi\}_{\xi \in D_d,\Delta} \).

Given \( 0 \leq m \leq d \) and \( T := \langle v_1, v_2, v_3 \rangle \), we say that a domain point \( \xi_{ijk} \) is at a distance \( \text{dist}(\xi, v_i) = d - i \) from the vertex \( v_i \) and at a distance \( \text{dist}(\xi, e_j) = i \) from the edge \( e_j = \langle v_2, v_3 \rangle \) opposite to \( v_i \). Furthermore, we refer to the set of domain points \( R_m(v_i) := \{ \xi_{d-m,j,m-j} \}_{j=0}^m \) as the ring of radius \( m \) around the vertex \( v_i \). We refer to the set \( D_m^T(v_i) := \bigcup_{n=0}^m R^T_n(v_i) \) as the disk of radius \( m \) around the vertex \( v_i \). The rings and disks around \( v_2 \) and \( v_3 \) are defined similarly. If \( v \) is a vertex of \( \Delta \) with triangles \( T_1, \ldots, T_k \) attached to it, then the ring and the disk of radius \( m \) around \( v \) are defined by \( R_m(v) = \bigcup_{i=1}^k R^T_i(v) \) and \( D_m(v) = \bigcup_{i=1}^k D^T_m(v) \), respectively.

Suppose now that \( S \) is a linear subspace of \( S^0_d(\Delta) \) defined by enforcing some set of smoothness conditions across the edges of the triangulation \( \Delta \). Then a determining set \( \mathcal{D} \) for \( S \) is a subset \( M \) of the set of domain points \( D_d,\Delta \) such that if we set the B-coefficients \( c_\xi \) of some spline \( s \in S \) to zero for all \( \xi \in M \), then \( s \equiv 0 \). If \( M \) is a determining set for a spline space \( S \) and \( M \) has the smallest cardinality among all possible determining sets for \( S \), then we call \( M \) a minimal determining set (MDS) for \( S \). It is known that \( M \) is a MDS for \( S \) if and only if every spline \( s \in S \) is uniquely determined by its set of B-coefficients \( \{c_\xi\}_{\xi \in M} \).

An MDS \( M \) is called local provided that there is an integer \( \ell \) such that for every \( \xi \in D_{d,\Delta} \cap T \) and every \( T \in \Delta \), the B-coefficient \( c_\xi \) of a spline \( s \in S \) is a linear combination of \( \{c_\eta\}_{\eta \in \Gamma_\xi} \) where \( \Gamma_\xi \) is a subset of \( M \) with \( \Gamma_\xi \subset \text{star}^\ell(T) \). Here \( \text{star}^\ell(T) := \text{star}(\text{star}^{\ell-1}(T)) \) for \( \ell \geq 2 \), where if \( U \) is a cluster of triangles, \( \text{star}(U) = \text{star}^1(U) \) is the set of all triangles which have a nonempty intersection with some triangle of \( U \). Moreover, \( M \) is said to be stable provided that there is a constant \( K \) depending only on \( d \) and the smallest angle in \( \Delta \) such that \( |c_\xi| \leq K \max_{\eta \in \Gamma_\xi} |c_\eta| \), for all \( \xi \in D_{d,\Delta} \).

We say that a spline \( s \in S^0_d(\Delta) \) is \( C^\rho \) smooth at the vertex \( v \) provided that all polynomials \( s|_T \) such that \( T \) is a triangle with vertex at \( v \) have common partial derivatives up to order \( \rho \) at the point \( v \). In this case we write \( s \in C^\rho(v) \).

Smoothness across an edge is described with the help of smoothness functionals defined as follows. Let \( T = \langle v_1, v_2, v_3 \rangle \) and \( \tilde{T} = \langle v_4, v_3, v_2 \rangle \) be two adjoining triangles which share the edge \( e = \langle v_2, v_3 \rangle \), and let \( c_{ijk} \) and \( \tilde{c}_{ijk} \) be the coefficients of the B-representations of \( s_T \) and \( s_{\tilde{T}} \), respectively. Then for any \( n \leq m \leq d \), let \( \tau^n_{e,m} \) be the linear functional defined on \( S^0_d(\Delta) \) by

\[
\tau^n_{e,m} := \tilde{c}_{n,m-n,d-m} - \sum_{i+j+k=n} c_{i,j+d,m,k+m-n} B_{ijk}^n(v_4).
\]

In terms of these linear functionals, the condition that \( s \) be \( C^r \) smooth across the edge \( e \) is equivalent to

\[
\tau^n_{e,m} = 0, \quad n \leq m \leq d, \quad 0 \leq n \leq r.
\]

### 3 Refinable spaces of $C^2$ piecewise quintics

Let \( \Omega \) be a bounded connected polygonal domain in \( \mathbb{R}^2 \). Suppose that some initial regular triangulation \( \Delta_0 \) of \( \Omega \) is given. Beginning with \( \Delta_0 \) we construct a sequence
\(\{\Delta_n\}_{n=0}^{\infty}\) of triangulations of \(\Omega\) by uniform refinement, that is \(\Delta_{n+1}\) is obtained from \(\Delta_n\) by subdividing any triangle \(T\) of \(\Delta_n\) into four equal subtriangles by joining the midpoints of the three edges with each other, as in Figure 1 (left). The uniform refinement of a single triangle \(T\) will be denoted \(T_U\).

Starting from \(\{\Delta_n\}_{n=0}^{\infty}\) we introduce further subdivisions by splitting each triangle of \(\Delta_n\) into twelve triangles by joining the midpoints of the three edges with each other and with the opposite vertices. This Powell-Sabin-12 split \(T_{PS12}\) of a single triangle \(T\) is illustrated in Figure 1 (right). Clearly, \(T_{PS12}\) is a refinement of \(T_U\). The triangulation obtained from \(\Delta_n\) by applying the Powell-Sabin-12 split to each triangle will be denoted \(\Delta_n^*\). An important observation is that \(\Delta_{n+1}^*\) is a refinement of \(\Delta_n^*\), in the sense that \(\Delta_{n+1}^*\) can be obtained from \(\Delta_n^*\) by subdividing its triangles.

![Figure 1: Uniform refinement \(T_U\) and Powell-Sabin-12 split \(T_{PS12}\) of a triangle.](image)

For each \(n = 0, 1, \ldots\), we denote by \(V_n, E_n, \bar{E}_n\) and \(W_n\) the sets of all vertices, edges, interior edges and midpoints of edges of \(\Delta_n\), respectively. Given a triangle \(T \in \Delta_n\), we denote the vertices of \(T_{PS12}\) by \(v_1, v_2, v_3, w_1, w_2, w_3, u_1, u_2, u_3\) and \(v_T\) as shown in Figure 1. We refer to the edges of the form \([v_i, u_i]\) as type-1 edges, to edges of the form \([u_i, v_T]\) as type-2 edges and to edges of the form \([w_i, v_T]\) as type-3 edges. For \(i = 1, 2, 3\), we write \(E_i^n\) for the set of all edges of \(\Delta_n^*\) of type-i.

We set

\[\tilde{V}_n = \emptyset, \quad \tilde{V}_n = (V_n \cap \text{Int } \Omega) \setminus \tilde{V}_0, \quad n = 1, 2, \ldots.\]

Then

\[V_n \setminus \tilde{V}_n = V_0 \cup (V_n \cap \partial \Omega).\]

For any \(v \in \bigcup_{n \in \mathbb{N}} \tilde{V}_n\), let \(n_v := \min\{n : v \in \tilde{V}_n\}\). Clearly, there is a unique edge \(e_v\) of \(\Delta_{n_v-1}\), with adjacent triangles \(T^+_v, T^-_v \in \Delta_{n_v-1}\), such that \(v\) lies at the midpoint of \(e_v\). Since \(W_n \cap \text{Int } \Omega \subset \tilde{V}_{n+1}\), the triangles \(T^+_w, T^-_w \in \Delta_n\) are well defined for any \(w \in W_n \cap \text{Int } \Omega\).

For \(n = 0, 1, \ldots\), let \(S_5^2(\Delta_n^*)\) denote the space of \(C^2\) quintic piecewise polynomials, i.e.

\[S_5^2(\Delta_n^*) := \{s \in C^2(\Omega) : s|_T \in \mathbb{P}_5 \text{ for all } T \in \Delta_n^*\}.\]
We consider the subspace $S_n$ of $S^2_0(\Delta_n^*)$ defined by

$$S_n = \{ s \in S^2_0(\Delta_n^*) : $$

(i) $s \in C^3(v)$ for all $v \in V_0 \cup (V_n \cap \partial \Omega)$ and all $v \in W_n \cap \partial \Omega$,

(ii) $s|_{T^+_v} \in C^3(v)$, $s|_{T^-_v} \in C^3(v)$ for all $v \in \hat{V}$ and all $v \in W_n \cap \text{Int} \Omega$, and

(iii) $s$ is $C^3$ across all edges in $\mathcal{E}_n^1 \cup \mathcal{E}_n^2 \cup \mathcal{E}_n^3$.

One crucial property of the spaces $S_n$ is their nestedness.

**Theorem 1.** The spaces $S_n$, $n = 0, 1, \ldots$ are nested, that is,

$$S_n \subset S_{n+1}, \quad n = 0, 1, 2, \ldots$$

**Proof.** Let $n \geq 1$. We suppose $s \in S_{n-1}$ and show that $s \in S_n$. If $v \in V_0 \cup (V_n \cap \partial \Omega)$, then $v \in V_0 \cup (V_{n-1} \cap \partial \Omega)$ or $v \in W_{n-1} \cap \partial \Omega$, so $s \in C^3(v)$ by Condition (i) in the definition of $S_n$. It is also clear that $s \in C^3(v)$ for $v \in W_n \cap \partial \Omega$ since $v$ lies in the interior of a boundary edge of $\Delta_{n-1}^*$. If $v \in \hat{V}$, then either $v \in \hat{V}_{n-1}$, or $n_v = n$ and $v \in W_{n-1} \cap \text{Int} \Omega$, $T^+_v$, $T^-_v \in \Delta_{n-1}$ and $v$ lies at the midpoint of the common edge $e_v$ of these two triangles. By Condition (ii) in both cases $s|_{T^+_v}$, $s|_{T^-_v} \in C^3(v)$ as required. If $v \in W_n \cap \text{Int} \Omega$, then $n_v = n + 1$ and $T^+_v$, $T^-_v \in \Delta_n$, whereas $v$ lies at the midpoint of the common edge $e_v \in \Delta_n$ of these two triangles. Moreover, for a triangle $T \in \Delta_{n-1}$, $v$ is either the midpoint of the edges $\langle v_1, w_2 \rangle$, $\langle v_1, w_3 \rangle$, $\langle v_2, w_1 \rangle$, $\langle v_2, w_3 \rangle$, $\langle v_3, w_1 \rangle$, $\langle v_3, w_2 \rangle$ or the vertices $u_1, u_2, u_3$ of $T_{PS12}$. In the first case, $s|_{T^+_v}$, $s|_{T^-_v} \in C^3(v)$ because $v$ is not a vertex of $\Delta_{n-1}^*$. In the second case, $s|_{T^+_v}$, $s|_{T^-_v} \in C^3(v)$ since $s$ is $C^3$ across type 1 and type 2 edges, $e \in \mathcal{E}_{n-1}^1 \cup \mathcal{E}_{n-1}^2$. If $e \in (\mathcal{E}_{n-1}^1 \cup \mathcal{E}_{n-1}^2 \cup \mathcal{E}_{n-1}^3)$ then either $e$ is (a part of) an edge $\tilde{e} \in (\mathcal{E}_{n-1}^1 \cup \mathcal{E}_{n-1}^2 \cup \mathcal{E}_{n-1}^3)$ since $\Delta_{n}^*$ is a refinement of $\Delta_{n-1}^*$ or $e$ lies in the interior of some triangle $T \in \Delta_{n-1}^*$. In both cases, $s$ is $C^3$ across $e$ by Condition (iii). \[\Box\]

We now want to generate a stable local MDS for $S_n$. For each $v \in \hat{V}$, let $e_v$ in $\Delta_{n-1}^*$ be the unique edge with adjacent triangles $T^+_v$, $T^-_v \in \Delta_{n-1}^*$ such that $v$ lies at the midpoint of $e_v$. For each $v \in V_n$, we choose a triangle $\hat{T}_v \in \Delta_{n}^*$ with vertex at $v$. If $v \in V_n$, we assume that $\hat{T}_v \subset T^+_v$ and we choose another triangle $\hat{T}_v = \langle v, u, w \rangle \in \Delta_{n}^*$ attached to $v$ such that $\hat{T}_v \subset T^-_v$ and an edge of $\hat{T}_v$ is a part of $e_v$. We now set $M_v := D_3(v) \cap \hat{T}_v$ for any $v \in V_n$, and $\hat{M}_v := M_v \cup \xi_3^{T^+_v}$ for any $v \in \hat{V}$. The set $M_v$ is illustrated in Figure 2.

Furthermore, for each edge $e$ of $\Delta_n$, let $v_{Te}$ be the barycentre of a triangle $T_e$ in $\Delta_n$ attached to $e$, let $w_e$ be the midpoint of $e$, let $T^+_e = \langle v_T, w_e, u \rangle$ be one of the triangles in $\Delta_{n}^*$ attached to the edge $\langle w_e, v_T \rangle$. For each $w \in W_n \cap \text{Int} \Omega$, let $T^+_w$ and $T^-_w$ be two triangles in $\Delta_n$ attached to the edge $e = \langle v_1, v_2 \rangle$ in $\Delta_n$, such that $w$ is the midpoint of $e$, that is $w = w_e$. Let $v_{Te}^+$ be the barycentre of $T^+_e$ and let $\hat{T}_e \subset T^+_e$ be some triangle in $\Delta_{n}^*$ attached to the edge $\langle w_e, v_{Te}^+ \rangle$ of type 3. Let $\hat{T}_e = \langle w_e, u, v_1 \rangle \subset T^-_e$ be one of the triangles in $\Delta_{n}^*$ with vertex $w_e$ such that one of its edges is a part of $e$. Let $\hat{M}_e = \{T_3^3, T_3^3, T_3^3, T_3^3, T_3^3, T_3^3 \} \cup \{T_3^3 \}$. The domain points corresponding to $\hat{M}_e$ are shown in Figure 3.
Theorem 2. The dimension of $S_n$ is given by
\[ \dim S_n = 10 \# \mathcal{V}_n + \# \tilde{\mathcal{V}}_n + 4 \# \mathcal{E}_n + \# \tilde{\mathcal{E}}_n. \] (4)
Moreover, the set
\[ M_n = \bigcup_{v \in \mathcal{V}_n} M_v \cup \bigcup_{v \in \mathcal{V}_6} \tilde{M}_v \cup \bigcup_{e \in \mathcal{E}_n} M_e \cup \bigcup_{e \in \mathcal{E}_6} \tilde{M}_e \]
is a stable local minimal determining set for \( S_n \).

The proof of this theorem will be given in Section 4.

By restricting to a single Powell-Sabin-12 split \( T_{PS12} \), we consider the space \( S(T_{PS12}) \) defined by
\[
S(T_{PS12}) = \{ s \in S_3^2(T_{PS12}) : \\
\begin{align*}
&\forall i = 1, 2, 3, \quad s \in C^3(v_i) \\
&\forall i = 1, 2, 3, \quad s \in C^3(w_i) \\
&\text{s is } C^3 \text{ across the segment } \langle v_i, u_i \rangle, \langle u_i, v_T \rangle, \langle w_i, v_T \rangle, \forall i = 1, 2, 3 \},
\end{align*}
\]
where \( v_i, u_i, w_i \) are as in Figure 1. Clearly, \( S(T_{PS12}) = S_0 \) if \( \Delta_0 \) consists of just one triangle.

Let \( T_1 = \langle v_1, w_3, u_1 \rangle, T_2 = \langle v_2, w_1, u_2 \rangle, T_3 = \langle v_3, w_2, u_3 \rangle, T_4 = \langle v_T, w_2, u_1 \rangle, T_5 = \langle v_T, w_3, u_2 \rangle, T_6 = \langle v_T, w_1, u_3 \rangle \), and let
\[ M_v = \bigcup_{i=1}^3 (D_3(v_i) \cap T_i), \quad M_e = \bigcup_{i=4}^6 \{ \xi_{T_i}^{T_1}, \xi_{T_i}^{T_2}, \xi_{T_i}^{T_3}, \xi_{T_i}^{T_4}, \xi_{T_i}^{T_5}, \xi_{T_i}^{T_6} \}. \]

Theorem 2 specialised to the case of \( S(T_{PS12}) \) gives the following corollary, see Figure 4 for an illustration.

**Corollary 3.** The dimension of \( S(T_{PS12}) \) is 42. Moreover, the set \( M = M_v \cup M_e \) is a stable minimal determining set for \( S(T_{PS12}) \).

**Remark 1.** The spaces \( S(T_{PS12}) \) can be used to define non-nested \( C^2 \) macro-element spaces which in fact coincide with the spaces of \( C^2 \) Powell-Sabin-12 macro-elements constructed in [11] when the \( v_T \) is the barycentre of \( T \). Note that our definition of \( S(T_{PS12}) \) is simpler than the corresponding space \( S_2(T_{PS12}) \) in [11].

## 4 Proof of Theorem 2

We start by providing two auxiliary results.

Let \( T_{PS6} \) be the Powell-Sabin-6 split of the triangle \( T = \langle w_1, w_2, w_3 \rangle \) which lies inside the Powell-Sabin-12 split in Figure 1 and is shown separately in Figure 5 for convenience. Recall that \( u_i \) is the midpoint of the edge opposite to \( w_i \) for \( i = 1, 2, 3 \), and \( v_T = (w_1 + w_2 + w_3)/3 \) is the barycentre of \( T \). We consider the space \( S_3^2(T_{PS6}) \) of all \( C^3 \) piecewise quintics on \( T_{PS6} \).

Let \( T_1 = \langle v_T, w_1, u_3 \rangle, T_2 = \langle v_T, w_2, u_1 \rangle \) and \( T_3 = \langle v_T, w_3, u_2 \rangle \), and let
\[ M = \bigcup_{i=1}^3 (D_3(w_i) \cap T_i), \]
see Figure 6, where the points in \( M \) are marked with filled circles.
Lemma 4. The dimension of $S^3_5(T_{PS6})$ is 30. Moreover, the above set $M$ is a stable minimal determining set for $S^3_5(T_{PS6})$.

Proof. The dimension of $S^3_5(T_{PS6})$ is easily obtained by [8, Theorem 9.3]. Let us show that $M$ is a minimal determining set for $S^3_5(T_{PS6})$. For each $i = 1, 2, 3$, we use the $C^3$ smoothness at $w_i$ to uniquely and stably compute the coefficients corresponding to all domain points in $D_3(w_i) \setminus M$ by [8, Lemma 5.10].

Next, for each edge $e_1 = \langle w_2, w_3 \rangle$, $e_2 = \langle w_1, w_3 \rangle$, $e_3 = \langle w_1, w_2 \rangle$ of $T$, we use the $C^3$ smoothness across the edge $\langle v_T, u_1 \rangle$, $\langle v_T, u_2 \rangle$, $\langle v_T, u_3 \rangle$, respectively, to determine the
coefficients corresponding to domain points in the set
\[ E_2(e_i) := \{ \xi : \text{dist}(\xi, e_i) \leq 2, \xi \notin D_3(w_i) \cup D_3(w_{i+1}) \}, \quad i = 1, 2, 3. \]
(These coefficients are indicated by squares in Figure 6.) The \( C^3 \) smoothness across the edge \( \langle v_T, u_i \rangle \) gives three smoothness conditions involving these coefficients which uniquely determine them as solutions of the corresponding linear system. For example, the barycentric coordinates of \( w_3 \) relative to \( \langle w_2, u_1, v_T \rangle \) are given by \((-1, 2, 0)\) since \( u_1 \) is the midpoint of the edge \( \langle w_2, w_3 \rangle \). Hence, the three smoothness conditions across the edge \( \langle v_T, u_1 \rangle \) involving the coefficients on the edge \( \langle w_2, w_3 \rangle \) are given by
\[
\begin{align*}
C_{23} &= -C_7 + 2C_6, \\
C_{27} &= C_{15} - 4C_7 + 4C_6, \\
C_{30} &= -C_{18} + 6C_{15} - 12C_7 + 8C_6,
\end{align*}
\]
where the coefficients \( C_i \) of a spline \( s \in S^3_3(T_{PS6}) \) are numbered as in Figure 6. By solving this linear system of equations with respect to \( C_7, C_{23} \) and \( C_6 \), we get
\[
\begin{align*}
C_7 &= -\frac{1}{4}C_{18} - \frac{1}{4}C_{30} + C_{15} + \frac{1}{2}C_{27}, \\
C_{23} &= -\frac{1}{4}C_{18} - \frac{1}{4}C_{30} + \frac{1}{2}C_{15} + C_{27}, \\
C_6 &= -\frac{1}{4}C_{18} - \frac{1}{4}C_{30} + \frac{3}{4}C_{15} + \frac{3}{4}C_{27}.
\end{align*}
\]
Similarly, we obtain
\[
\begin{align*}
C_8 &= -\frac{1}{4}C_{17} - \frac{1}{4}C_{29} + C_{14} + \frac{1}{2}C_{26}, \quad C_{22} = -\frac{1}{4}C_{17} - \frac{1}{4}C_{29} + \frac{1}{2}C_{14} + C_{26}, \\
C_5 &= -\frac{1}{4}C_{17} - \frac{3}{4}C_{29} + \frac{3}{2}C_{14} + \frac{3}{2}C_{26}, \quad C_9 = -\frac{1}{4}C_{16} - \frac{1}{4}C_{28} + C_{13} + \frac{1}{2}C_{25}, \\
C_{21} &= -\frac{1}{4}C_{16} - \frac{1}{4}C_{28} + \frac{3}{2}C_{13} + C_{25}, \quad C_4 = -\frac{1}{4}C_{16} - \frac{1}{4}C_{28} + \frac{3}{4}C_{13} + \frac{3}{4}C_{25}.
\end{align*}
\]
The other coefficients indicated by squares in Figure 6 can be found in the same way.

By taking into account the \( C^3 \) smoothness condition across the edges \( \langle w_1, v_T \rangle, \langle w_2, v_T \rangle, \langle w_3, v_T \rangle \), we compute the yet unknown coefficients corresponding to the domain points on the rings \( R_3(w_1), R_4(w_2) \) and \( R_4(w_3) \), respectively, compare [8, Lemma 2.30]. These coefficients are indicated by diamonds in Figure 6. For example, the barycentric coordinates of \( u_1 \) relative to \( \langle u_3, w_3, v_T \rangle \) are \((-1, \frac{1}{2}, \frac{1}{2})\), and hence the three smoothness conditions across the edge \( \langle w_2, v_T \rangle \) involving the coefficients in the ring \( R_4(w_2) \) are given by

\[
C_{10} = -C_{31} + \frac{1}{2}C_{12} + \frac{3}{2}C_{11}, \quad C_9 = C_{32} - C_{35} - 3C_{31} + \frac{1}{4}C_{16} + \frac{3}{2}C_{12} + \frac{9}{8}C_{11}, \\
C_8 = -C_{33} + \frac{3}{2}C_{34} + \frac{9}{8}C_{32} - \frac{9}{8}C_{36} - \frac{9}{8}C_{35} - \frac{9}{8}C_{31} + \frac{9}{8}C_{37} + \frac{9}{8}C_{16} + \frac{9}{8}C_{12} + \frac{9}{8}C_{11}.
\]

By solving the linear system involving the above equations, we get

\[
C_{10} = \frac{5}{3}C_9 + \frac{2}{3}C_{32} - \frac{2}{3}C_{35} + \frac{1}{3}C_{16} - \frac{4}{9}C_8 - \frac{4}{9}C_{34} + \frac{2}{3}C_{34} - \frac{1}{3}C_{12} + \frac{1}{18}C_{37} - \frac{1}{3}C_{36}, \\
C_{11} = \frac{5}{3}C_9 + \frac{2}{3}C_{32} + \frac{1}{3}C_{16} - \frac{16}{27}C_8 - \frac{16}{27}C_{33} + \frac{4}{9}C_{34} - \frac{4}{9}C_{36} + \frac{2}{3}C_{37} - \frac{4}{9}C_{35}, \\
C_{31} = \frac{2}{3}C_9 + \frac{2}{3}C_{32} - \frac{1}{3}C_{35} + \frac{1}{3}C_{12} - \frac{4}{9}C_8 - \frac{4}{9}C_{33} + \frac{2}{3}C_{34} - \frac{1}{3}C_{36} + \frac{1}{18}C_{37} + \frac{1}{3}C_{16}.
\]

By using \( C^1 \) smoothness across the edges \( \langle u_1, v_T \rangle, \langle u_2, v_T \rangle, \langle u_3, v_T \rangle \) we compute the remaining undetermined coefficients corresponding to the domain points at distances three and four from \( \langle w_1, w_2 \rangle, \langle w_2, w_3 \rangle, \langle w_3, w_1 \rangle \). These coefficients are marked by stars in Figure 6. For instance, since that the coefficients \( C_{19} \) and \( C_{20} \) are already known, we compute \( C_2 \) and \( C_3 \) using the formulas

\[
C_2 = \frac{1}{2}C_{11} + \frac{1}{2}C_{19}, \quad C_3 = \frac{1}{2}C_{10} + \frac{1}{2}C_{20}.
\]

Finally, the only remaining undetermined coefficient at \( v_T \), marked by a triangle in Figure 6, can be computed by using for example the univariate \( C^1 \) smoothness condition along the line \( \langle u_2, w_2 \rangle \), which gives

\[
C_1 = \frac{2}{3}C_{11} + \frac{1}{3}C_{38}.
\]

We have shown that \( M \) is a determining set for \( S_3^3(T_{PS6}) \). The set \( M \) is minimal since its cardinality is equal to the dimension of \( S_3^3(T_{PS6}) \). The stability of \( M \) is obvious in view of [8, Lemma 5.10] and the above explicit formulas. \( \square \)

**Remark 2.** The space \( S_3^3(T_{PS6}) \) coincides with the space \( S_2(T_{PS6}) \) of [8, Theorem 7.9], where the vertex \( v_T \) is placed at the barycentre of \( T \).

**Lemma 5.** Let \( \Delta \) be the triangulation shown in Figure 7 with six vertices \( v_1, \ldots, v_6 \), where \( v_4 = (v_3 + v_5)/2 \), \( v_2 = (3v_1 + v_3)/4 \) and \( v_6 = (3v_1 + v_5)/4 \). Let \( T = \langle v_1, v_4, v_6 \rangle \). Let \( M = D_1(v_3) \cup D_1(v_5) \cup M_e \subset D_3, \Delta \), where

\[
M_e = \{ \xi_{1,2,0}^T, \xi_{2,1,0}^T, \xi_{3,0,0}^T, \xi_{2,0,1}^T \}.
\]

Then \( M \) is a stable minimal determining set for the space \( \mathbb{P}_3 \) of cubic polynomials regarded as a subspace of \( S_3^3(\Delta) \).
Figure 7: Triangulation of Lemma 5, where the minimal determining set $M$ is indicated by filled circles.

Proof. The lemma follows from a more general statement given in [11, Lemma 4.1]. We provide a (somewhat different) proof in order to work out explicit formulas for the $B$-coefficients. We see that $\# M = \dim \mathbb{P}_3 = 10$. Hence we only need to show that if we set the coefficients of $s \in \mathbb{P}_3$ corresponding to $\xi \in M$, then all other coefficients are stably determined. Suppose the coefficients of $s$ are numbered as in Figure 7. Then using the $C^1$ smoothness across the common edge $\langle v_1, v_4 \rangle$ of two triangle $T_1 = \langle v_2, v_4, v_1 \rangle$ and $T_2 = \langle v_6, v_1, v_4 \rangle$, where the barycentric coordinates of $v_6$ relative to $T_1$ are given by $(-1, \frac{1}{2}, \frac{3}{2})$, we obtain

$$C_{13} = \frac{1}{2}C_{12} + \frac{3}{2}C_1 - C_2.$$  

By using the univariate $C^1$, $C^2$ and $C^3$ smoothness conditions along the edge $\langle v_1, v_3 \rangle$, where the barycentric coordinates of $v_3$ relative to $\langle v_1, v_2 \rangle$ are given by $(-3, 4)$ and solving the linear system involving the three equations, we obtain

$$C_{14} = -\frac{9}{16}C_1 - \frac{1}{48}C_{28} + \frac{3}{16}C_{13} + \frac{1}{12}C_{27},$$

$$C_{23} = -\frac{27}{16}C_1 - \frac{1}{16}C_{28} + \frac{3}{16}C_{13} + \frac{1}{2}C_{27},$$

$$C_{22} = -\frac{27}{32}C_1 - \frac{1}{32}C_{28} + \frac{27}{16}C_{13} + \frac{3}{16}C_{27}.$$  

By symmetry, similar formulas hold for $C_3, C_4$ and $C_5$, and by using the univariate $C^1$, $C^2$ and $C^3$ smoothness conditions along the edge $\langle v_3, v_5 \rangle$ and solving the corresponding linear system, we get the formulas

$$C_{25} = -\frac{1}{4}C_{28} - \frac{1}{4}C_7 + C_{26} + \frac{1}{2}C_8,$$

$$C_{18} = -\frac{1}{4}C_{28} - \frac{1}{4}C_7 + \frac{1}{2}C_{26} + C_8,$$

$$C_{19} = -\frac{1}{4}C_{28} - \frac{1}{4}C_7 + \frac{3}{4}C_{26} + \frac{3}{4}C_8.$$
In the next step we compute \( C_{20} \) and \( C_{17} \) by \( C^1 \) smoothness conditions across edges \( \langle v_2, v_4 \rangle \) and \( \langle v_4, v_6 \rangle \), respectively,

\[
C_{20} = \frac{1}{4}C_{25} + \frac{3}{16}C_{16}, \quad C_{17} = \frac{1}{4}C_{18} + \frac{3}{16}C_{16}.
\]

By the \( C^1 \) and \( C^2 \) smoothness conditions across the edge \( \langle v_1, v_4 \rangle \), as well as the \( C^2 \) smoothness conditions across the edges \( \langle v_2, v_4 \rangle \) and \( \langle v_4, v_6 \rangle \), we obtain the system of equations

\[
\begin{align*}
C_{11} &= -C_{15} + \frac{1}{2}C_{16} + \frac{3}{2}C_{12}, \\
C_{10} &= C_{21} - C_{20} - 3C_{15} + \frac{1}{4}C_{19} + \frac{3}{2}C_{16} + \frac{3}{4}C_{12}, \\
C_{26} &= 9C_{12} - 24C_{15} + 16C_{21}, \\
C_{8} &= 16C_{12} - 24C_{11} + 9C_{10},
\end{align*}
\]

which can be solved with respect to \( C_{10}, C_{11}, C_{15} \) and \( C_{21} \) to give

\[
\begin{align*}
C_{10} &= -\frac{1}{4}C_{8} + C_{12} + \frac{13}{4}C_{16} + \frac{11}{4}C_{19} - \frac{15}{4}C_{20} - \frac{3}{4}C_{26}, \\
C_{11} &= -\frac{2}{21}C_{8} + \frac{25}{21}C_{12} + \frac{9}{14}C_{16} + \frac{3}{14}C_{19} - \frac{3}{7}C_{20} - \frac{3}{56}C_{26}, \\
C_{15} &= \frac{2}{21}C_{8} + \frac{11}{21}C_{12} - \frac{1}{7}C_{16} - \frac{3}{14}C_{19} + \frac{6}{7}C_{20} - \frac{3}{56}C_{26}, \\
C_{21} &= \frac{1}{7}C_{8} + \frac{1}{8}C_{12} - \frac{3}{14}C_{16} - \frac{9}{28}C_{19} + \frac{9}{7}C_{20} - \frac{3}{56}C_{26}.
\end{align*}
\]

Finally, using \( C^1 \) smoothness across the edges \( \langle v_4, v_6 \rangle \) and \( \langle v_2, v_4 \rangle \), we get

\[
C_{9} = -3C_{11} + 4C_{10}, \quad C_{24} = -3C_{15} + 4C_{21},
\]

which completes the proof that \( M \) is a determining set for \( \mathbb{P}_3 \). The stability is again obvious in view of the explicit formulas used. \( \square \)

Proof of Theorem 2. To see that \( M_n \) is a stable minimal determining set, we show that we can set the coefficients \( \{ c_\xi \}_{\xi \in M_n} \) of a spline \( s \in S_n \) to arbitrary values, and that all other coefficients of \( s \) are then uniquely and stably determined.

First, we show how the coefficients in \( \mathcal{D}_{3,\Delta^*} \setminus M_n \) can be computed. For each \( v \in \mathcal{V}_n \setminus \tilde{\mathcal{V}}_n \), using \( C^3 \) smoothness conditions at \( v \) and the coefficients corresponding to \( M_v \), we can uniquely compute the coefficients of \( s \) corresponding to all domain points in \( D_3(v) \) by [8, Lemma 5.10]. For each \( v \in \tilde{\mathcal{V}}_n \), using the \( C^3 \) smoothness of \( s|_{T^+_v} \) at \( v \) and the coefficients in \( M_v \cap T^+_v \), we compute the coefficients of \( s \) corresponding to all domain points in \( D_3(v) \cap T^+_v \). By using \( C^2 \) smoothness across the common edge \( e_v \) of \( T^+_v \) and \( T^-_v \), we can compute the coefficients corresponding to \( D_2(v) \cap T^-_v \). Then by \( C^3 \) smoothness at \( v \) inside \( T^-_v \) and the coefficient corresponding to the domain point \( \xi^{T^-_v}_{\delta_2,\delta_3,0} \) inside \( T^-_v \) we can compute the remaining coefficients corresponding to all domain points in \( R_3(v) \cap T^-_v \).

For each \( e = \langle u, v \rangle \in \mathcal{E}_n \setminus \tilde{\mathcal{E}}_n \), using the coefficients corresponding to \( M_e \), we now apply Lemma 5 to determine the coefficients of \( s \) corresponding to domain points in the disk \( D_3(w_e) \), where \( w_e \) is the midpoint of \( e \). Due to the \( C^4 \) smoothness at \( w_e \), we can regard the coefficients of \( s \) in the disk as coefficients of a polynomial \( g \) of degree 3. Lemma 5 ensures that we can set the coefficients of \( s \) corresponding to domain points in
$M_e$ to arbitrary values, and that all coefficients corresponding to the remaining domain points in $D_3(w_e)$ are uniquely and stably determined.

For each $e = \langle u, v \rangle \in \mathcal{E}_n$, using the $C^3$ smoothness of $s|_{T^e}$ at the midpoint $w_e$ of $e$ and the coefficients corresponding to \{\$\xi_{1,2,0}^3, \xi_{2,3,0}^3, \xi_{2,2,1}^3, \xi_{1,4,0}^3\}$ inside $T^e_+$ we can compute the coefficients of $s$ corresponding to domain points in the disk $D_3(w_e) \cap T^e_+$ by Lemma 5 as described previously. Using the $C^2$ smoothness across the common edge $e$ of $T^e_+$ and $T^e_-$ we can compute the coefficients corresponding to $D_3(w_e) \cap T^e_-$. Then using the $C^3$ smoothness condition supported inside $T^e_-$ and the coefficient corresponding to the point $\xi_{2,3,0}^T$, we can compute the remaining coefficients corresponding to domain points in $R_3(w_e) \cap T^e_-$. For each type-1 edge $e = \langle u, v \rangle$, by taking account of the $C^3$ smoothness across the edge $e = \langle u, v \rangle$ we can now compute the three central coefficients in the ring $R_4(v)$. Note that in practice these coefficients can be more conveniently computed by using $C^1$ smoothness conditions across the edges of the form $\langle w_i, w_{i+1} \rangle$, see Remark 3.

We now show that the coefficients corresponding to the remaining domain points are uniquely determined. These remaining domain points lie inside triangles of the form $T = \langle w_1, w_2, w_3 \rangle$, where $w_i \in \mathcal{W}_n$. Let $T_{PS6}$ be the Powell-Sabin-6 split of $T$, see Figure 5. We have already determined all coefficients corresponding to domain points in the disks $D_3(w_i)$ for $i = 1, 2, 3$. Now we can apply Lemma 4 to uniquely and stably determine all coefficients of $s$ corresponding to the remaining domain points in $T$.

We have thus shown that $M$ is a determining set for $S_n$. To complete the proof, we need to show that the six central $C^1, C^2$ smoothness conditions across the edges $\langle w_i, w_{i+1} \rangle$, $i = 1, 2, 3$ are satisfied. Indeed, all other smoothness conditions are either used in the above computation or are satisfied in view of Lemmas 4 and 5.

To check these conditions we will only look at the section $\langle v_1, w_3, v_T, w_2 \rangle$ of the triangle $T = \langle v_1, v_2, v_3 \rangle$ as shown in Figure 8, where we indicate the domain points of the $5 \times 7$ grid around $u_1$ by double integer indices $(i, j)$ with the origin $(0,0)$ at $u_1$ and the row of indices with $j = 0$ on the edge $\langle w_2, w_3 \rangle$. The coefficient corresponding to $(i, j)$ is denoted by $C_{i,j}$. All smoothness conditions that need verification are supported within this grid. The smoothness conditions on the other two sections of the triangle can be checked in the same way.

Let $(\gamma, \beta, \delta)$ be the barycentric coordinates of $v_T$ relative to $\langle v_1, w_3, u_1 \rangle$. Thus $\beta = 0$ and $v_T = \gamma v_1 + \delta u_1$ where $\gamma = -\frac{1}{3}$ and $\delta = \frac{4}{3}$. We first write down the known $C^1, C^2$ and $C^3$ smoothness conditions in rows $j = -2, -1, 0, 1, 2$ of the grid,

\[
\begin{align*}
C_{3,j} &= 8C_{0,j} - 12C_{-1,j} + 6C_{-2,j} - C_{-3,j}, \\
C_{2,j} &= 4C_{0,j} - 4C_{-1,j} + C_{-2,j}, \\
C_{1,j} &= 2C_{0,j} - C_{-1,j}.
\end{align*}
\]

By solving this linear system for $C_{-1,j}$, $C_{0,j}$, $C_{1,j}$, we obtain for $j = -2, -1, 0, 1, 2$

\[
\begin{align*}
C_{-1,j} &= \frac{1}{4}(2C_{2,j} - C_{3,j} + 4C_{-2,j} - C_{-3,j}), \\
C_{0,j} &= \frac{1}{4}(3C_{2,j} - C_{3,j} + 3C_{-2,j} - C_{-3,j}), \\
C_{1,j} &= C_{2,j} - \frac{1}{4}C_{3,j} + \frac{3}{4}C_{-2,j} - \frac{1}{4}C_{-3,j}.
\end{align*}
\]

We write down the four known $C^1$ smoothness conditions across row 0 of the grid

\[
C_{i,-1} = \gamma C_{i,1} + \delta C_{i,0}, \quad i = -3, -2, 2, 3.
\]
By replacing $C_{-3,-1}, C_{-2,-1}, C_{2,-1}, C_{3,-1}$ in (5)–(7) with expressions in (8) and collecting the terms with coefficients $\gamma$ and $\delta$, we obtain
\[
C_{i,-1} = \gamma C_{i,1} + \delta C_{i,0}, \quad i = -1, 0, 1,
\]
which confirms the three remaining $C^1$ smoothness conditions across row 0. Similarly, by replacing $C_{3,2}, C_{2,2}, C_{-2,2}, C_{-3,2}$ in (5)–(7) with expressions in the four known $C^2$ smoothness conditions across row 0,
\[
C_{i,2} = \gamma^2 C_{i,0} + 2\gamma\delta C_{i,-1} + \delta^2 C_{i,-2}, \quad i = -3, -2, 2, 3,
\]
we verify the three remaining $C^2$ smoothness conditions across row 0,
\[
C_{i,2} = \gamma^2 C_{i,0} + 2\gamma\delta C_{i,-1} + \delta^2 C_{i,-2}, \quad i = -1, 0, 1.
\]

We have shown that $M_n$ is a stable local minimal determining set for $S_n$. Hence, the dimension of $S_n$ is equal to the cardinality of $M_n$, which is easily seen to be the number in (4).

**Remark 3.** Note that in practice the three central coefficients in the ring $R_4(v_1)$ are more conveniently computed by using $C^1$ smoothness conditions across the edge $\langle w_3, w_2 \rangle$ rather than by $C^1$, $C^2$ and $C^3$ smoothness conditions across $\langle v_1, u_1 \rangle$ according to the above proof. Indeed, in the notation of Figure 8 these coefficients are given by
\[
C_{-1,1} = 4C_{-1,0} - 3C_{-1,-1}, \quad C_{0,1} = 4C_{0,0} - 3C_{0,-1}, \quad C_{1,1} = 4C_{1,0} - 3C_{1,-1}.
\]
5 A nodal minimal determining set for $S_n$

As usual for macro-element spaces, we provide a stable nodal minimal determining set for $S_n$ and an error bound for the corresponding Hermite interpolation operator.

Recall that a linear functional $\lambda$ is called a nodal functional provided that $\lambda f$ is a combination of values and/or derivatives of $f$ at some point $\eta$. A collection $\mathcal{N} = \{\lambda\}^N_{i=1}$ is called a nodal determining set for a spline space $S$ if $\lambda s = 0$ for all $\lambda \in \mathcal{N}$ implies $s \equiv 0$. Moreover, $\mathcal{N}$ is called a nodal minimal determining set (NMDS) for $S$ if there is no smaller nodal determining set. We refer to [8, Section 5.9] for further details on nodal determining sets.

Let $(u_x, u_y)$ and $(v_x, v_y)$ be the Cartesian coordinates of $u$ and $v$, respectively. Then the directional derivative of $s$ at $(x, y) \in T$ with respect to the (directed) edge $e$ is given by

$$D_e s(x, y) = (v_x - u_x) D_x s(x, y) + (v_y - u_y) D_y s(x, y).$$

Let $e^\perp$ be the directed segment obtained rotating $e$ ninety degrees in the counterclockwise direction. We write $D_{e^\perp} s$ for the directional derivative of $s$ associated with $e^\perp$. The linear functional evaluating at $\xi \in \Omega$ any function $f$ continuous at $\xi$ will be denoted by $\delta_\xi$.

**Lemma 6.** Let $\Delta$ be the triangulation shown in Figure 7, where $v_4 = (v_3 + v_5)/2$, $v_2 = (3v_1 + v_3)/4$ and $v_6 = (3v_1 + v_5)/4$. The set

$$N = N_{v_3} \cup N_{v_4} \cup N_{v_5}$$

is a nodal determining set for $\mathbb{P}_3$, where

1) $N_{v_3} = \{\delta_{v_3}, \delta_{v_3} D_x, \delta_{v_3} D_y\}$,

2) $N_{v_5} = \{\delta_{v_5}, \delta_{v_5} D_x, \delta_{v_5} D_y\}$,

3) $N_{v_4} = \{\delta_{v_4} D_{v_1}, \delta_{v_4} D_{v_1}^2, \delta_{v_4} D_{v_2} D_{v_1}, \delta_{v_4} D_{v_1}^3\}$,

with $e_1 := \langle v_1, v_4 \rangle$ and $e_2 := \langle v_4, v_5 \rangle$.

**Proof.** It is clear that the cardinality of $N$ is equal to the dimension of $\mathbb{P}_3$. Thus to prove that $N$ is a nodal minimal determining set, we just need to show that given the values of $\{\lambda s\}_{\lambda \in \mathcal{N}}$ all B-coefficients of $s \in \mathbb{P}_3$ can be determined. Suppose the coefficients of $s \in \mathbb{P}_3$ are numbered as in Figure 7. Using the data $\{\delta_{v_3} s, \delta_{v_3} D_x s, \delta_{v_3} D_y s\}$ at $v_3$ we can compute the coefficients $C_{26}, C_{27}$ and $C_{28}$ by [8, Theorem 2.19]. Similarly, using the data $\{\delta_{v_5} s, \delta_{v_5} D_x s, \delta_{v_5} D_y s\}$ at $v_5$ we compute the coefficients $C_6, C_7$ and $C_8$. Using the data $\{\delta_{v_4} D_{v_1} s, \delta_{v_4} D_{v_1}^2 s, \delta_{v_4} D_{v_1}^3 s\}$ we can compute the coefficients $C_{16}, C_{12}$ and $C_1$ by [8, Lemma 2.20], that is, using the formulas

$$\begin{align*}
\delta_{v_4} D_{v_1} s &= -3C_{19} + 3C_{16}, \\
\delta_{v_4} D_{v_1}^2 s &= 6C_{19} - 12C_{16} + 6C_{12}, \\
\delta_{v_4} D_{v_1}^3 s &= -6C_{19} + 18C_{16} - 18C_{12} + 6C_1,
\end{align*}$$

where $C_{19}$ can be computed using the three univariate smoothness conditions along the edge $\langle v_3, v_5 \rangle$ as in the proof of Lemma 5.
Let $e = \langle v_4, v_6 \rangle$. Since $v_6 - v_4 = \frac{1}{4}(v_6 - v_4) + \frac{3}{4}(v_1 - v_4)$, according to [8, (2.36)], we can use the data $\delta_{v_4}D_{e_2}D_{e_1}^2s$ to compute the coefficient $C_2$ from the relation

$$\frac{1}{4}\delta_{v_4}D_{e_2}D_{e_1}^2s + \frac{3}{4}\delta_{v_4}D_{e_1}^3s = \delta_{v_4}D_{e}D_{e_1}s = -6C_{19} + 12C_{16} + 6C_{12} + 12C_{11} - 6C_{17} - 6C_{2},$$

where the coefficients $C_{11}$ and $C_{17}$ are computed as in the proof of Lemma 5.

At this point we have determined all coefficients corresponding to domain points in the minimal determining set $M$ of Lemma 5, and it follows from that lemma that all other coefficients are also determined. □

**Theorem 7.** The set

$$N_n = \bigcup_{v \in V_n \setminus \check{V}_n} N_v \cup \bigcup_{v \in \check{V}_n} \check{N}_v \cup \bigcup_{e \in E_n \setminus \check{E}_n} N_e \cup \bigcup_{e \in \check{E}_n} \check{N}_e$$

is a nodal minimal determining set for $S_n$, where

1) $N_v = \{\delta_v D_{\alpha}^a D_{\beta}^b, \ 0 \leq \alpha + \beta \leq 3\}$,

2) $\check{N}_v = \{\delta_v D_{\alpha}^a D_{\beta}^b, \ 0 \leq \alpha + \beta \leq 3, \beta \leq 2\} \cup \{\delta_v^+ D_{\alpha}^a D_{\beta}^b, \delta_v^- D_{\alpha}^a D_{\beta}^b\}$,

3) $N_e = \{\delta_{w_e} D_{e_+}^{a_+} D_{e_-}^{b_+}, \delta_{w_e} D_{e_+}^{a_-} D_{e_-}^{b_-}, \delta_{w_e} D_{e_+}^{a_0} D_{e_-}^{b_0}\}$,

4) $\check{N}_e = \{\delta_{w_e} D_{e_+}^{a_+}, \delta_{w_e} D_{e_+}^{a_-}, \delta_{w_e} D_{e_+}^{a_0}, \delta_{w_e} D_{e_-}^{a_0}, \delta_{w_e} D_{e_-}^{a_-}\}$,

where $\delta_v^\pm f := \delta_v (f|_{T_v^\pm})$ and $\delta_w^\pm f := \delta_w (f|_{T_v^\pm})$, and $w_e$ denotes the midpoint of the edge $e$.

**Proof.** It is clear that the cardinality of $N_n$ is equal to the dimension of $S_n$ as given in (4). Thus to prove that $N_n$ is a nodal minimal determining set, we just need to show that the values of $\{\lambda_s\}_{\lambda \in N_n}$ all B-coefficients of $s \in S_n$ can be determined.

For every vertex $v \in V_n \setminus \check{V}_n$, we can compute all coefficients corresponding to domain points in the disk $D_3(v)$ directly from the data in $N_v$ by [8, Theorem 2.19].

For every vertex $v \in \check{V}_n$, we can compute all the coefficients of $s$ corresponding to domain points in the disk $D_3(v)$ from the data in $\check{N}_v$. That is, using the data $\{\delta_v D_{\alpha}^a D_{\beta}^b, \ 0 \leq \alpha + \beta \leq 3, \beta \leq 2\} \cup \{\delta_v^+ D_{\alpha}^a D_{\beta}^b\}$, we can compute the coefficients corresponding to domain points in $D_3(v) \cap T_v^+$ by [8, Theorem 2.19]. Then using the coefficients corresponding to domain points in $D_3(v) \cap T_v^+$ and $C^2$ smoothness conditions across the edge $e_v$, we can compute the coefficients corresponding to domain points in $D_2(v) \cap T_v^-$. Now using the data $\{\delta_v^- D_{\alpha}^a D_{\beta}^b\}$ and $C^3$ smoothness conditions inside $T_v^-$, we can compute all the remaining coefficients corresponding to domain points in $R_3(v) \cap T_v^-$. Given an edge $e = \langle v', v'' \rangle$ in $E_n \setminus \check{E}_n$, let $w_e$ be its midpoint. We now compute all coefficients of $s$ corresponding to domain points in $D_3(w_e)$. By the $C^3$ smoothness at $w_e$, as in the proof of Theorem 2, these coefficients can be regarded as the coefficients of a polynomial $g$ of degree 3. Hence, it follows from Lemma 6 that all B-coefficients in $D_3(w_e)$ are determined by the known B-coefficients in the sets $D_3(w_e) \cap D_3(v')$ and $D_3(w_e) \cap D_3(v'')$ and the nodal data in $N_e$.  

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Given an edge \( e \) in \( \mathcal{E}_n \), we can compute the coefficients of \( s \) corresponding to the domain points in the disk \( D_3(w_e) \) from the data in \( \mathcal{N}_e \). That is, using the data \( \{ \delta_{w_e} D_{e^{-}} s, \delta_{w_e} D_{e^+} D_{e^{-}} D_{x} s, \delta_{w_e} D_{e^+} D_{e^{-}} D_{y} s, \delta_{w_e} D_{e^+} D_{e^{-}} D_{x} D_{y} s \} \), we can compute the B-coefficients corresponding to \( D_3(w_e) \cap T^+_e \) using the same argument as above. Then using the coefficients corresponding to domain points in \( D_3(w_e) \cap T^-_e \), and \( C^2 \) smoothness conditions across the edge \( e \) we can compute all the coefficients corresponding to domain points in \( D_3(w_e) \cap T^-_e \). Finally, using \( \{ \delta_{w_e} D_{e^+} D_{x} s \} \) and \( C^3 \) smoothness conditions inside \( T^-_e \) we can compute all the remaining coefficients corresponding to domain points in \( R_3(w_e) \cap T^-_e \). At this point we have determined all coefficients corresponding to domain points in the minimal determining set \( M_n \) of Theorem 2, and it follows from that theorem that all other coefficients are also determined. \[ \square \]

**Corollary 8.** The set \( N = N_v \cup N_e \) is a nodal minimal determining set for \( S(T_{P_{S_{12}}} \Omega) \), where

1) \( N_v = \bigcup_{i=1}^{3} \{ \delta_{w_i} D_x^\alpha D_y^\beta, 0 \leq \alpha + \beta \leq 3 \} \),

2) \( N_e = \bigcup_{i=1}^{3} \{ \delta_{w_i} D_{e^+}, \delta_{w_i} D_{e^-} D_{x}, \delta_{w_i} D_{e^-} D_{y}, \delta_{w_i} D_{e^-} D_{x} D_{y} \} \),

\( v_i \) are the three vertices of \( T \), \( e_1 := \langle v_1, v_2 \rangle \), \( e_2 := \langle v_2, v_3 \rangle \), \( e_3 := \langle v_3, v_1 \rangle \) and \( w_e \) denotes the midpoint of \( e_i \).

By Theorem 7 for any function \( f \in C^3(\Omega) \) and any \( n = 0, 1, \ldots \), there exists a unique spline \( s_n(f) \in S_n \) that solves the Hermite interpolation problem

\[
\lambda s = \lambda f, \quad \lambda \in N_n
\]

The following error bound follows immediately by [8, Theorem 5.26] if we take into account that the uniform refinement used to generate the triangulation \( \Delta_n \) halves the diameters of the triangles and that stability of the nodal minimal determining set \( N_n \) can be verified by usual arguments, see e.g. [11, p. 724].

**Theorem 9.** For every \( f \in C^r(\Omega) \), with \( 3 \leq r \leq 6 \),

\[
|f - s_n(f)|_{W^k_\infty(\Omega)} \leq \frac{K}{2^{n(r-k)}} |f|_{W^k_\infty(\Omega)}
\]

for all \( 0 \leq k < r \), where \( K \) depends only on the maximum diameter and the smallest angle of the triangles of the initial triangulation \( \Delta_0 \), and \( | \cdot |_{W^k_\infty(\Omega)} \) denotes the standard Sobolev seminorm on \( \Omega \).

**Remark 4.** It can be easily checked that the nodal determining sets of Theorem 7 are nested, that is, \( N_n \subset N_{n+1}, n = 0, 1, 2, \ldots \). This fact may be useful for certain multilevel algorithms, see e.g. [1].

**Remark 5.** In developing the macroelement spaces of this paper, we have used P. Alfeld’s software for examining determining set for the spline spaces, available from [http://www.math.utah.edu/~alfeld](http://www.math.utah.edu/~alfeld).
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References


