Approximation by sums of piecewise linear polynomials

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June 9, 2014

Abstract

We present two partitioning algorithms that allow a sum of piecewise linear polynomials over a number of overlaying convex partitions of the unit cube Ω in \( \mathbb{R}^d \) to approximate a function \( f \in W^3_p(\Omega) \) with the order \( N^{-6/(2d+1)} \) in \( L^p \)-norm, where \( N \) is the total number of cells of all partitions, which makes a marked improvement over the \( N^{-2/d} \) order achievable on a single convex partition. The gradient of \( f \) is approximated with the order \( N^{-3/(2d+1)} \). The first algorithm creates \( d \) convex partitions and relies on the knowledge of the eigenvectors of the average Hessians of \( f \) over the cells of an auxiliary uniform partition, whereas the second algorithm with \( \binom{d+1}{2} \) convex partitions is independent of \( f \). In addition, we also give an \( f \)-independent partitioning algorithm for a sum of \( d \) piecewise constants that achieves the approximation order \( N^{-2/(d+1)} \).

1 Introduction

Let \( \Omega = (0, 1)^d \), \( d \geq 2 \). A finite set \( \Delta \) of subdomains \( \omega \) of \( \Omega \) (called cells) is said to be a partition of \( \Omega \) if \( \omega \cap \omega' = \emptyset \) when \( \omega \neq \omega' \), and \( \sum_{\omega \in \Delta} |\omega| = |\Omega| \), where \( |\omega| \) denotes the Lebesgue measure (\( d \)-dimensional volume) of \( \omega \). A partition is convex if each cell \( \omega \) is a convex domain. The cardinality of a finite set \( D \) is denoted \( |D| \), so that \( |\Delta| \) stands for the number of cells \( \omega \) in the partition \( \Delta \).

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Given a partition $\Delta$, the linear space of piecewise polynomials of order $k$ with respect to it is defined by

$$S_k(\Delta) = \left\{ \sum_{\omega \in \Delta} q_\omega \chi_\omega : q_\omega \in \Pi^d_k \right\},$$

where $\Pi^d_k$, $k \geq 1$, is the space of polynomials of total degree $< k$ in $d$ variables. The error of the best $L_p$-approximation of a function $f \in L_p(\Omega)$ from $S_k(\Delta)$,

$$E_k(f, \Delta)_p := \inf_{s \in S_k(\Delta)} \|f - s\|_p, \quad 1 \leq p \leq \infty,$$

can be computed if the errors $E_k(f)_{L_p(\omega)} := \inf_{q \in \Pi^d_k} \|f - q\|_{L_p(\omega)}$ of the best polynomial approximations of $f$ on all $\omega \in \Delta$ are known. Indeed,

$$E_k(f, \Delta)_p = \begin{cases} \left( \sum_{\omega \in \Delta} E_k(f)_{L_p(\omega)} \right)^{1/p} & \text{if } p < \infty, \\ \max_{\omega \in \Delta} E_k(f)_{L_\infty(\omega)} & \text{if } p = \infty. \end{cases} \quad (1)$$

For a system $P = \{ \Delta^{(1)}, \ldots, \Delta^{(n)} \}$ of several overlaying partitions of $\Omega$, we consider the space of sums of piecewise polynomials

$$S_k(P) = \left\{ \sum_{\nu=1}^n \sum_{\omega \in \Delta^{(\nu)}} q_{\nu, \omega} \chi_\omega : q_{\nu, \omega} \in \Pi^d_k \right\}.$$

Thus, a function $s$ in $S_k(P)$ is the sum of $n$ piecewise polynomials $s = \sum_{\nu=1}^n s_\nu$ with $s_\nu \in S_k(\Delta^{(\nu)})$, $\nu = 1, \ldots, n$. We set $|P| := \sum_{\nu=1}^n |\Delta^{(\nu)}|$ and denote the best approximation error from $S_k(P)$ by

$$E_k(f, P)_p := \inf_{s \in S_k(P)} \|f - s\|_p, \quad 1 \leq p \leq \infty.$$

Given a function $f$, we consider piecewise polynomial approximations of $f$ on suitably designed partitions. Standard uniform type partitions deliver piecewise polynomial approximations with the order

$$E_k(f, \Delta)_p = O(|\Delta|^{-k/d}), \quad |\Delta| \to \infty, \quad (2)$$

if $f$ belongs to the Sobolev space $W^k_p(\Omega)$, as follows from the Bramble-Hilbert lemma, see for example [3].

It is shown in [1, Theorem 2] that the approximation order of piecewise constants $E_1(f, \Delta)_\infty = O(|\Delta|^{-1/d})$ cannot be improved even assuming infinite differentiability of $f$ if the partitions are isotropic. One thus has to use anisotropic partitions if smoothness should pay off in convergence rate.
A simple algorithm suggested in [1, 2] (see Algorithm 1 and Theorem 1 below) delivers an improved approximation order $E_1(f, \Delta)_p = \mathcal{O}(|\Delta|^{-2/(d+1)})$ of piecewise constants on suitable anisotropic convex partitions if $f \in W^2_p(\Omega)$. Here, the unit cube is first subdivided uniformly into $m^d$ subcubes (macro-cells), each of which is then split anisotropically into $m$ slices (micro-cells). (Note that in the case $d = 2$ the order $E_1(f, \Delta)_p = \mathcal{O}(|\Delta|^{-2/3})$ has been obtained earlier in [4] by a different method.) Moreover, [2, Theorem 2] shows that $|\Delta|^{-2/(d+1)}$ is the saturation order of piecewise constant approximation on convex partitions in the sense that it cannot be further improved for any $f \in C^2(\Omega)$ whose Hessian is positive definite at some point. Nevertheless, [2, Theorem 3] suggests that this phenomenon is restricted to piecewise constants, as the saturation order of piecewise linear approximations on convex partitions is $|\Delta|^{-2/d}$, that is the same as on the isotropic partitions.

In this paper we show that for $k = 2$ the approximation order $E_2(f, \Delta)_p = \mathcal{O}(|\Delta|^{-2/d})$ in (2) can be improved to $E_2(f, \mathcal{P})_p = \mathcal{O}(|\mathcal{P}|^{-6/(2d+1)})$ if $f \in W^3_p(\Omega)$ by using a sum of piecewise linear polynomials with respect to a system $\mathcal{P}$ of $d$ convex polyhedral partitions of $\Omega$ (Algorithm 3 and Theorem 3). Moreover, the approximation of the gradient of $f$ improves to $\mathcal{O}(|\mathcal{P}|^{-3/(2d+1)})$ from the standard estimate $\mathcal{O}(|\Delta|^{-1/d})$ for piecewise linear polynomials on a single partition.

In addition, we show that the sums of $d$ piecewise constants on suitable fixed, $f$-independent partitions can be used to obtain the same approximation order $E_1(f, \mathcal{P})_p = \mathcal{O}(|\mathcal{P}|^{-2/(d+1)})$ (Algorithm 2 and Theorem 2), whereas Algorithm 1 relies on the knowledge of the average gradients of $f$ on the macro-cells. Similarly, the sums of $\binom{d+1}{2}$ piecewise linear polynomials on $f$-independent partitions can be used to obtain the approximation order $E_2(f, \mathcal{P})_p = \mathcal{O}(|\mathcal{P}|^{-6/(2d+1)})$ (Algorithm 4 and Theorem 4), whereas Algorithm 3 employs the average Hessians of $f$ on the macro-cells.

The results presented here are based on Chapter 4 of the thesis of the second named author [5].

The paper is organised as follows. Section 2 is devoted to the piecewise constant approximation, where after recalling the algorithm suggested in [1, 2] we present our new result for the sums of piecewise constants with fixed splitting directions of the macro-cells, whereas in Sections 3 and 4 we describe the two algorithms for the sums of piecewise linear polynomials.

In what follows we will use the following version of the Sobolev seminorm

$$|f|_{W^k_p(\omega)} := \sum_{|\alpha|=n} \left\| \frac{\partial^n f}{\partial x^\alpha} \right\|_{L_p(\omega)}, \quad |\alpha| := \alpha_1 + \cdots + \alpha_d \quad \text{for } \alpha \in \mathbb{Z}_+^d,$$

and recall that if $\omega \subset \mathbb{R}^d$ is a bounded convex domain and $f|_{\omega} \in W^k_p(\omega)$, then
there exists a polynomial \( q \in \Pi_k^d \) such that [3]
\[
|f - q|_{W^r_p(\omega)} \leq \rho_{d,k} \diam^{k-r}(\omega)|f|_{W^k_p(\omega)}, \quad r = 0, \ldots, k,
\] (3)
where \( \rho_{d,k} \) denotes a positive constant depending only on \( d \) and \( k \) [3]. In view of Lemma 1, (3) implies in particular the Poincaré inequality
\[
\|f - f_\omega\|_{L_p(\omega)} \leq \rho_d \diam(\omega)\|\nabla f\|_{L_p(\omega)}, \quad f \in W^1_p(\omega),
\] (4)
with a constant \( \rho_d \) depending only on \( d \), where \( f_\omega := |\omega|^{-1} \int_\omega f(x) \, dx \) and
\[
\|\nabla f\|_{L_p(\omega)} := \left( \sum_{k=1}^d |D_{x_k} f|^2 \right)^{1/2}, \quad D_{x_k} f := \frac{\partial f}{\partial x_k}.
\]
Note that \( \|f_\omega - c\|_{L_p(\omega)} \leq \|f - c\|_{L_p(\omega)} \) for any constant \( c \), and hence \( \|f - f_\omega\|_{L_p(\omega)} \leq 2E_1(f)_{L_p(\omega)} \). We prefer to use (4) rather than (3) when \( k = 1 \) because explicit values or estimates of the optimal constant in (4) are known for \( p = 1, 2, \infty \), see a discussion and references in [1, Section 2].

Lemma 1. For any \( 1 \leq p \leq \infty \),
\[
\|\nabla f\|_{L_p(\omega)} \leq |f|_{W^1_p(\omega)} \leq d^{\max\left\{ \frac{1}{2}, 1 - \frac{1}{p} \right\}} \|\nabla f\|_{L_p(\omega)}. \tag{5}
\]

Proof. By the inequality between discrete 2- and 1-norms, and triangle inequality, we have
\[
\|\nabla f\|_{L_p(\omega)} \leq \left( \int_\omega \left( \sum_{k=1}^d |D_{x_k} f(x)|^p \right)^{1/p} \, dx \right)^{1/p} \leq \sum_{k=1}^d \left( \int_\omega |D_{x_k} f(x)|^p \, dx \right)^{1/p},
\]
which shows the first inequality in (5). The second one is obtained as follows. By the inequality between arithmetic and \( p \)-power means,
\[
|f|_{W^1_p(\omega)} \leq d^{1 - \frac{1}{p}} \left( \sum_{k=1}^d \int_\omega |D_{x_k} f(x)|^p \, dx \right)^{1/p},
\]
which completes the proof if \( p = 2 \). If \( p > 2 \), then \( |f|_{W^1_p(\omega)} \leq d^{1 - \frac{1}{p}} \|\nabla f\|_{L_p(\omega)} \) follows by the inequality between \( p \)- and 2-norms. If \( p < 2 \), then the inequality between \( p \)- and 2-means leads to \( |f|_{W^1_p(\omega)} \leq d^2 \|\nabla f\|_{L_p(\omega)} \). \( \square \)
2 Sums of piecewise constants

The following algorithm for piecewise constant approximation with optimal approximation order $|\Delta|^{-2/(d+1)}$ on convex polyhedral partitions has been suggested in [1, 2].

**Algorithm 1 ([2]).** Assume $f \in W^1_p(\Omega), \Omega = (0,1)^d$. Split $\Omega$ into $N_1 = m^d$ cubes $\omega_1, \ldots, \omega_{N_1}$ of edge length $1/m$, with $m \in \mathbb{Z}_+$. Then split each $\omega_i$ into $N_2$ slices $\omega_{ij}$, $j = 1, \ldots, N_2$, by equidistant hyperplanes orthogonal to the average gradient $g_i := |\omega_i|^{-1} \int_{\omega_i} \nabla f(x) \, dx$ on $\omega_i$. Set $\Delta = \{ \omega_{ij} : i = 1, \ldots, N_1, j = 1, \ldots, N_2 \}$. Clearly, $|\Delta| = N_1 N_2$ and each $\omega_{ij}$ is a convex polyhedron with at most $2(d+1)$ facets.

**Theorem 1 ([2]).** Assume that $f \in W^2_p(\Omega), \Omega = (0,1)^d$, for some $1 \leq p \leq \infty$. For any $m = 1, 2, \ldots$, generate the partition $\Delta_m$ by using Algorithm 1 with $N_1 = m^d$ and $N_2 = m$. Then

$$E_1(f, \Delta_m)_p \leq C |\Delta_m|^{-2/(d+1)} (|f|_{W^1_p(\Omega)} + |f|_{W^2_p(\Omega)}),$$

(6)

where $C$ is a constant depending only on $d$.

The new algorithm will involve a system of $d$ convex polyhedral partitions independent of $f$.

**Algorithm 2.** Split $\Omega = (0,1)^d$ into $N_1 = m^d$, $m \in \mathbb{Z}_+$, cubes $\omega_1, \ldots, \omega_{N_1}$ of edge length $1/m$, whose edges are parallel to the coordinate axes. For each $\nu = 1, \ldots, d$, define $\Delta^{(\nu)}$ by splitting each $\omega_i$ into $N_2$ slices $\omega_{ij}^{(\nu)}$, $j = 1, \ldots, N_2$, by equidistant hyperplanes orthogonal to the $x_\nu$-axis. Set $\mathcal{P} = \{ \Delta^{(1)}, \ldots, \Delta^{(d)} \}$. Then $|\Delta^{(\nu)}| = N_1 N_2$ for all $\nu = 1, \ldots, d$ and $|\mathcal{P}| = d N_1 N_2$.

Partitions $\Delta^{(1)}, \Delta^{(2)}$ in the case $d = 2$ and $N_2 = m = 4$ are illustrated in Fig. 1. Note that each $\omega_{ij}^{(\nu)}$ is a $d$-dimensional box with its $\nu$-th dimension $\frac{1}{m N_2}$ and all other dimensions $\frac{1}{m}$.

**Theorem 2.** Assume that $f \in W^2_p(\Omega), \Omega = (0,1)^d$, for some $1 \leq p \leq \infty$. For any $m = 1, 2, \ldots$, generate the system of partitions $\mathcal{P}_m$ by using Algorithm 2 with $N_1 = m^d$ and $N_2 = m$. Then

$$E_1(f, \mathcal{P}_m)_p \leq C |\mathcal{P}_m|^{-2/(d+1)} (|f|_{W^1_p(\Omega)} + |f|_{W^2_p(\Omega)}),$$

(7)

where $C$ is a constant depending only on $d$. 


Figure 1: Partitions $\Delta^{(1)}, \Delta^{(2)}$ for piecewise constant approximation ($d = 2$, $N_2 = m = 4$).

**Proof.** For the sake of brevity we assume that $p < \infty$. The proof for $p = \infty$ is the same except that at the appropriate places integrals have to be replaced by the $L_\infty$-norm.

We first introduce an auxiliary piecewise linear approximation of $f$. For each $i = 1, \ldots, N_1$, let

$$\ell_i = c_i + \sum_{\nu=1}^{d} \ell_{i,\nu},$$

where

$$c_i = |\omega_i|^{-1} \int_{\omega_i} f(x) \, dx, \quad \ell_{i,\nu}(x) = a_{i,\nu}(x_{\nu} - x_{i,\nu}),$$

with

$$a_{i,\nu} := |\omega_i|^{-1} \int_{\omega_i} D_{x_{\nu}} f(x) \, dx, \quad \nu = 1, \ldots, d,$$

and $(x_{i,1}, \ldots, x_{i,d})$ denotes the barycenter of $\omega_i$. We set

$$\ell := \sum_{i=1}^{N_1} \ell_i \chi_{\omega_i}.$$  

Since

$$\int_{\omega_i} \ell_{i,\nu}(x) \, dx = 0, \quad \nu = 1, \ldots, d,$$

and diam$(\omega_i) \leq \frac{\sqrt{d}}{m}$, we deduce by the Poincaré inequality and (5),

$$\left\| \left( f - \sum_{\nu=1}^{d} \ell_{i,\nu} \right) - c_i \right\|_{L_p(\omega_i)} \leq \rho_d \text{diam}(\omega_i) \left\| \nabla f - \sum_{\nu=1}^{d} \nabla \ell_{i,\nu} \right\|_{L_p(\omega_i)}$$

$$\leq \frac{\sqrt{d} \rho_d}{m} \sum_{\nu=1}^{d} \| D_{x_{\nu}} f - a_{i,\nu} \|_{L_p(\omega_i)}.$$
The Poincaré inequality and (5) also imply
\[
\|D_{x^\nu} f - a_{i,\nu}\|_{L^p(\omega_i)} \leq \rho_d \text{diam}(\omega_i) \|\nabla (D_{x^\nu} f)\|_{L^p(\omega_i)}
\]
\[
\leq \frac{d \rho_d}{m} \sum_{\mu=1}^d \|D_{x^\mu} f\|_{L^p(\omega_i)},
\]
where we set \(D_{x^\mu} f := D_{x^\mu} D_{x^\nu} f\). It follows that
\[
\|f - \ell_i\|_{L^p(\omega_i)} \leq \frac{d \rho_d^2}{m^2} \sum_{\nu,\mu=1}^d \|D_{x^\mu} f\|_{L^p(\omega_i)} = \frac{2d \rho_d^2}{m^2} |f|_{W^2_2(\omega_i)}.
\]

Hence,
\[
\|f - \ell\|_p = \left(\sum_{i=1}^{N_1} \|f - \ell_i\|^p_{L^p(\omega_i)}\right)^{\frac{1}{p}} \leq \frac{2d \rho_d^2}{m^2} |f|_{W^2_2(\omega)}.
\] (8)

For each \(i, \nu\), let \([x_j^0, x_j^1]\) be the projection of the \(\nu\)-th edge of \(\omega_{ij}^{(\nu)}\) on the \(x^\nu\)-axis, \(j = 1, \ldots, N_2\). We now replace \(\ell_{i,\nu}\) by the piecewise constant function
\[
s_{i,\nu} = \sum_{j=1}^{N_2} b_j a_{i,\nu} \chi_{\omega_{ij}^{(\nu)}},
\]
where \(b_j = x_j^0 - x_{i,\nu}\), \(j = 1, \ldots, N_2\). Since \(x_j^1 - x_j^0 = \frac{1}{mN_2}\), we obtain
\[
\|\ell_{i,\nu} - s_{i,\nu}\|_{L^p(\omega_i)}^p = \sum_{j=1}^{N_2} \int_{\omega_{ij}^{(\nu)}} |a_{i,\nu}(x_{\nu} - x_{i,\nu} - b_j)|^p \, dx
\]
\[
= \frac{|a_{i,\nu}|^p}{m^{d-1}} \sum_{j=1}^{N_2} \int_0^{\frac{mN_2}{jN_2}} u^p \, du = \left(\frac{1}{mN_2}\right)^p \frac{|a_{i,\nu}|^p}{m^d(p + 1)}.
\]

Observe that
\[
\frac{|a_{i,\nu}|^p}{m^d} = |\omega_i| \frac{1}{|\omega_i|} \int_{\omega_i} |D_{x^\nu} f(x)|^p \, dx \leq \int_{\omega_i} |D_{x^\nu} f(x)|^p \, dx.
\]

By setting
\[
s = \sum_{i=1}^{N_1} \left(c_i + \sum_{\nu=1}^d s_{i,\nu}\right) \chi_{\omega_i},
\]
we obtain a function \( s \in S_1(P_m) \) satisfying
\[
\| \ell - s \|_p = \left( \sum_{i=1}^{N_1} \left( \sum_{\nu=1}^{d} (\ell_{i,\nu} - s_{i,\nu}) \right)^p \right)^{\frac{1}{p}} \\
\leq \left( \sum_{i=1}^{N_1} \left( \sum_{\nu=1}^{d} \| \ell_{i,\nu} - s_{i,\nu} \|_{L_p(\omega_i)} \right)^p \right)^{\frac{1}{p}} \\
\leq \sum_{\nu=1}^{d} \left( \sum_{i=1}^{N_1} \| \ell_{i,\nu} - s_{i,\nu} \|_{L_p(\omega_i)} \right)^{\frac{1}{p}},
\]
by the triangle inequalities for both integral and discrete \( p \)-norm. In view of the estimates given above,
\[
\| \ell - s \|_p \leq \frac{(p + 1)^{-1/p}}{mN_2} \sum_{\nu=1}^{d} \left( \sum_{i=1}^{N_1} \left( \frac{|a_{i,\nu}|^p}{m^d} \right)^{\frac{1}{p}} \right) \\
\leq \frac{1}{mN_2} \sum_{\nu=1}^{d} \left( \sum_{i=1}^{N_1} \int_{\omega_i} |D_{x,\nu} f(x)|^p \, dx \right)^{\frac{1}{p}} \\
= \frac{1}{mN_2^2} \max_{\omega} |f|_{W^{1,p}(\Omega)}. \tag{9}
\]
Since \( N_2 = m \) and \( m^{-2} = \left( \frac{\|P_m\|}{d} \right)^d \), the bound \( (7) \) with constant \( C = d^{d}^{\frac{d}{d+1}} \max \{2d\rho_d^2, 1\} \) is obtained by combining \( \text{(8)} \) and \( \text{(9)} \).

\section{Sums of piecewise linear polynomials}

In this section we approximate the function by using a sum of piecewise linear polynomials over several overlaying partitions of \( \Omega \).

**Algorithm 3.** Assume \( f \in W^{2,1}_1(\Omega), \Omega = (0,1)^d \). Split \( \Omega \) into \( N_1 = m^d \), \( m \in \mathbb{Z}_+ \), subcubes \( \omega_1, \ldots, \omega_{N_1} \) of edge length \( 1/m \), whose edges are parallel to the coordinate axes. For each \( i = 1, \ldots, N_1 \), let \( H_i \) be the average Hessian matrix of \( f \) over \( \omega_i \),
\[
H_i = \left[ \frac{1}{2|\omega_i|} \int_{\omega_i} D_{x,\nu\mu} f(x) \, dx \right]_{\nu,\mu=1,\ldots,d},
\]
and let \( \sigma_{i,\nu}, \nu = 1, \ldots, d \), be unit eigenvectors of \( H_i \). For each \( \nu = 1, \ldots, d \), define \( \Delta^{(\nu)} \) by splitting each \( \omega_i \) into \( N_2 \) slices \( \omega_{ij}^{(\nu)} \), \( j = 1, \ldots, N_2 \), by equidistant hyperplanes orthogonal to the eigenvector \( \sigma_{i,\nu} \). Set \( \mathcal{P} = \{ \Delta^{(1)}, \ldots, \Delta^{(d)} \} \).
Then \( |\Delta^{(\nu)}| = N_1N_2 \) for each \( \nu = 1, \ldots, d \) and \( |\mathcal{P}| = dN_1N_2 \).
Partitions $\Delta^{(1)}$, $\Delta^{(2)}$ of Algorithm 3 in the case $d = 2$ and $N_2 = m = 4$ are illustrated in Figure 2. The splitting directions on each subcube $\omega_i$ are orthogonal to one of the eigenvectors of the average Hessian $H_i$.

**Theorem 3.** Let $f \in W^3_p(\Omega)$, $\Omega = (0,1)^d$, for some $1 \leq p \leq \infty$. For any $m = 1, 2, \ldots$, generate the system of partitions $P_m$ by using Algorithm 3 with $N_1 = m^d$ and $N_2 = \lceil m^{\frac{d}{2}} \rceil$. Then there exists a sum of piecewise linear functions $s_m \in S_2(P_m)$ such that

$$E_2(f, P_m) \leq \|f - s_m\|_p \leq C_1|P_m|^{-6/(2d+1)}(|f|_{W^3_p(\Omega)} + |f|_{H^3_p(\Omega)}),$$  

$$|f - s_m|_{W^1_p(\Omega)} \leq C_2|P_m|^{-3/(2d+1)}(|f|_{W^3_p(\Omega)} + |f|_{H^3_p(\Omega)}),$$

where $C_1, C_2$ are constants depending only on $d$.

**Proof.** As in the proof of Theorem 2 we assume that $p < \infty$. The modifications needed in the case $p = \infty$ are obvious. Denote by $\Delta$ the partition of $\Omega$ into $N_1$ cubes $\omega_1, \ldots, \omega_{N_1}$ of edge length $1/m$. It follows from (3) that for each $i = 1, \ldots, N_1$ there exists a quadratic polynomial $q_i$ such that

$$\|f - q_i\|_{L^p(\omega_i)} \leq \rho_{d,3} \text{diam}^3(\omega_i) |f|_{W^3_p(\omega_i)} \leq \frac{d^2 \rho_{d,3}}{m^3} |f|_{W^3_p(\omega_i)},$$

$$|f - q_i|_{W^1_p(\omega_i)} \leq \rho_{d,3} \text{diam}^2(\omega_i) |f|_{W^3_p(\omega_i)} \leq \frac{d \rho_{d,3}}{m^2} |f|_{W^3_p(\omega_i)},$$

$$|f - q_i|_{W^2_p(\omega_i)} \leq \rho_{d,3} \text{diam}(\omega_i) |f|_{W^3_p(\omega_i)} \leq \frac{d \rho_{d,3}}{m} |f|_{W^3_p(\omega_i)}.$$

**Figure 2:** Partitions $\Delta^{(1)}$ (left) and $\Delta^{(2)}$ (right) obtained from Algorithm 3 ($d = 2$ and $N_2 = m = 4$).
We deduce from (12) that

$$\| f - \sum_{i=1}^{N_1} q_i \chi_{\omega_i} \|_p \leq \frac{d^2 \rho_{d,3}}{m^3} \| f \|_{W^3_p(\Omega)}. \quad (15)$$

Let $\tilde{q}_i(x) = x^T H_i x$ be the homogeneous quadratic polynomial whose Hessian matrix coincides with the average Hessian matrix $H_i$ of $f$ over $\omega_i$, that is,

$$D_{x^T x^\mu} \tilde{q}_i = |\omega_i|^{-1} \int_{\omega_i} D_{x^T x^\mu} f(x) \, dx, \quad \nu, \mu = 1, \ldots, d.$$ 

We can establish a relation between $q_i$ and $\tilde{q}_i$ as follows. By using the Poincaré inequality (4), together with (14), we obtain

$$\| D_{x^T x^\mu} (\tilde{q}_i - q_i) \|_{L^p(\omega_i)} \leq \| D_{x^T x^\mu} (\tilde{q}_i - f) \|_{L^p(\omega_i)} + \| D_{x^T x^\mu} (f - q_i) \|_{L^p(\omega_i)}$$

$$\leq \rho_d \text{diam}(\omega_i) \| \nabla (D_{x^T x^\mu} f) \|_{L^p(\omega_i)} + \rho_{d,3} \text{diam}(\omega_i) \| f \|_{W^3_p(\omega_i)}. \quad (16)$$

Let $i \in \{1, \ldots, N_1\}$ be fixed, and let $(x_{0,1}, \ldots, x_{0,d})$ denote the barycenter of $\omega_i$. Consider the linear polynomial $\tilde{\ell}_i$ defined by

$$\tilde{\ell}_i = \tilde{c}_i + \sum_{\nu=1}^d \tilde{\ell}_{i,\nu}, \quad \tilde{\ell}_{i,\nu} := \tilde{a}_{i,\nu} (x_{\nu} - x_{0,\nu}), \quad (17)$$

where

$$\tilde{a}_{i,\nu} = |\omega_i|^{-1} \int_{\omega_i} D_{x^\nu} (f - \tilde{q}_i)(x) \, dx, \quad \tilde{c}_i = |\omega_i|^{-1} \int_{\omega_i} (f - \tilde{q}_i)(x) \, dx. \quad (18)$$

Observe that $\int_{\omega_i} \tilde{\ell}_{i,\nu}(x) \, dx = 0$ for all $\nu = 1, \ldots, d$. Hence, by using (4), we obtain

$$\| f - \tilde{q}_i - \tilde{\ell}_i \|_{L^p(\omega_i)} = \| (f - \tilde{q}_i - \sum_{\nu=1}^d \tilde{\ell}_{i,\nu}) - \tilde{c}_i \|_{L^p(\omega_i)}$$

$$\leq \rho_d \text{diam}(\omega_i) \| \nabla (f - \tilde{q}_i - \sum_{\nu=1}^d \tilde{\ell}_{i,\nu}) \|_{L^p(\omega_i)}$$

$$\leq \rho_d \text{diam}(\omega_i) \| f - \tilde{q}_i - \sum_{\nu=1}^d \tilde{\ell}_{i,\nu} \|_{W^3_p(\omega_i)}. \quad (19)$$
We shall estimate the seminorm in the above inequality. To this end, observe that for each $\nu = 1, \ldots, d$, the Poincaré inequality and (18) yield

$$
\|D_{x,\nu} (f - \tilde{q}_i) - \sum_{\mu=1}^d \tilde{\ell}_{i,\mu} \|_{L_p(\omega_i)} = \|D_{x,\nu} (f - \tilde{q}_i) \|_{L_p(\omega_i)}
\leq \rho_d \text{diam}(\omega_i) \|\nabla (D_{x,\nu} (f - \tilde{q}_i))\|_{L_p(\omega_i)}
\leq \rho_d \text{diam}(\omega_i) |D_{x,\nu} (f - \tilde{q}_i)|_{W^1_p(\omega_i)}.
$$

(20)

Now, for each $\mu = 1, \ldots, d$, by virtue of the definition of $\tilde{q}_i$, the Poincaré inequality implies that

$$
\|D_{x,\mu x,\nu} (f - \tilde{q}_i) \|_{L_p(\omega_i)} \leq \rho_d \text{diam}(\omega_i) \|\nabla (D_{x,\mu x,\nu} f)\|_{L_p(\omega_i)}.
$$

(21)

Combining (19), (20) and (21) we obtain

$$
\|f - \tilde{q}_i - \tilde{\ell}_i\|_{L_p(\omega_i)} \leq \rho_d^3 \text{diam}(\omega_i)^3 |f|_{W^3_p(\omega_i)}.
$$

(22)

Using the above estimation, together with (12), yields

$$
\|q_i - \tilde{q}_i - \tilde{\ell}_i\|_{L_p(\omega_i)} \leq \|q_i - f\|_{L_p(\omega_i)} + \|f - \tilde{q}_i - \tilde{\ell}_i\|_{L_p(\omega_i)}
\leq d^2 \frac{(\rho_{d,3} + \rho_d^3)}{m^3} |f|_{W^3_p(\omega_i)}.
$$

(23)

Since $c_i$ is a constant, we have

$$
|f - \tilde{q}_i - \tilde{\ell}_i|_{W^3_p(\omega_i)} = |f - \tilde{q}_i - \sum_{\nu=1}^d \tilde{\ell}_{i,\nu}|_{W^3_p(\omega_i)} \leq \rho_d^2 \text{diam}(\omega_i)^2 |f|_{W^3_p(\omega_i)},
$$

by virtue of (20) and (21). Combining this with (13) implies

$$
|q_i - \tilde{q}_i - \tilde{\ell}_i|_{W^3_p(\omega_i)} \leq |q_i - f|_{W^3_p(\omega_i)} + |f - \tilde{q}_i - \tilde{\ell}_i|_{W^3_p(\omega_i)}
\leq d \frac{(\rho_{d,3} + \rho_d^3)}{m^3} |f|_{W^3_p(\omega_i)}.
$$

(24)

For each $i = 1, \ldots, N_1$, the Hessian matrix $H_i$ can be diagonalized into $H_i = U_i^T D_i U_i$ where $U_i$ is an orthogonal matrix

$$
U_i = [\sigma_{i,1} \cdots \sigma_{i,d}]^T,
$$

and $D_i$ is a diagonal matrix whose entries $\lambda_{i,1}, \ldots, \lambda_{i,d}$ are the eigenvalues of $H_i$. Then

$$
\tilde{q}_i = \lambda_{i,1} \ell_1^2 + \cdots + \lambda_{i,d} \ell_d^2,
$$

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where
\[ \ell_{\nu}(x) := \sigma_{i,\nu}^T x, \quad \nu = 1, \ldots, d, \]
are linear polynomials. We have
\[
|\lambda_{i,\nu}| \leq \|H_i\|_{\infty} = \frac{1}{2|\omega_i|} \max_{1 \leq \gamma \leq d} \sum_{\mu=1}^{d} \int_{\omega_i} \left| D_{\gamma,\nu} f(x) \right| dx \leq \frac{1}{2} |\omega_i|^{-1/p} \|f\|_{W_2^p(\omega_i)}. \]

Since \(|\omega_i| = m^{-d}\), it follows that
\[
|\lambda_{i,\nu}| \leq \frac{m^{d/p}}{2} |f|_{W_2^p(\omega_i)}, \quad \nu = 1, \ldots, d. \quad (25)
\]

Given \(i, \nu\) and \(j\), the set \(\omega_{ij}^{(\nu)}\) is contained between two hyperplanes \(\ell_{\nu}(x) = c_j\) and \(\ell_{\nu}(x) = c_j + w_{i,\nu} \frac{mN_2}{mN_2}\), where \(w_{i,\nu}\) denotes the width of the unit cube in the direction \(\sigma_{i,\nu}\). Clearly, \(1 \leq w_{i,\nu} \leq \sqrt{d}\). We set
\[
\bar{s}_{i,\nu} := \sum_{j=1}^{N_2} \lambda_{i,\nu} c_j (2\ell_{\nu} - c_j) \chi_{\omega_{ij}^{(\nu)}},
\]
\[
\bar{s}_i := \ell_i + \sum_{\nu=1}^{d} \bar{s}_{i,\nu}.
\]

Then by using the orthogonal change of variables \(y = \phi_U(x) := U_i x\) we obtain in view of (25),
\[
\|\lambda_{i,\nu} \ell_{\nu}^2 - \bar{s}_{i,\nu}\|_{L_p(\omega_{ij}^{(\nu)})}^p = \int_{\omega_{ij}^{(\nu)}} |\lambda_{i,\nu} (\ell_{\nu}(x) - c_j)^2|^p dx = \int_{\phi_U(\omega_{ij}^{(\nu)})} |\lambda_{i,\nu} (y_{\nu} - c_j)^2|^p dy
\]
\[
\leq \left( \frac{\sqrt{d}}{m} \right)^{d-1} \int_{c_j}^{c_j + w_{i,\nu} \frac{mN_2}{mN_2}} |\lambda_{i,\nu} (y_{\nu} - c_j)^2|^p dy_{\nu}
\]
\[
\leq \frac{d^{d/2}}{(2p + 1)m^d N_2} \left( \frac{d |\lambda_{i,\nu}|}{m^2 N_2^2} \right)^p
\]
\[
\leq \frac{d^{d/2}}{(2p + 1)N_2} \left( \frac{d}{2m^2 N_2^2} \right)^p |f|_{W_2^p(\omega_i)}^p, \quad (26)
\]

which implies
\[
|\lambda_{i,\nu} \ell_{\nu}^2 - \bar{s}_{i,\nu}|_{L_p(\omega_i)}^p \leq \sum_{j=1}^{N_2} |\lambda_{i,\nu} \ell_{\nu}^2 - \bar{s}_{i,\nu}|_{L_p(\omega_{ij}^{(\nu)})}^p \leq \frac{d^{d/2}}{2p + 1} \left( \frac{d}{2m^2 N_2^2} \right)^p |f|_{W_2^p(\omega_i)}^p.
\]
Hence
\[
\|\tilde{q}_i + \tilde{\ell}_i - \bar{s}_i\|_{L^p(\omega_i)} = \left\| \sum_{\nu=1}^{d} (\lambda_{i,\nu} \ell^2_{\nu} - \bar{s}_{i,\nu}) \right\|_{L^p(\omega_i)} \leq \left( \frac{d^{d/2}}{2p+1} \right)^{\frac{1}{p}} \frac{d^2}{2m^2 N_2^2} |f|_{W^2_p(\omega)} ,
\]
from which it immediately follows that
\[
\| \sum_{i=1}^{N_1} (\tilde{q}_i + \tilde{\ell}_i - \bar{s}_i) \chi_{\omega_i} \|_p = \left( \sum_{i=1}^{N_1} \|\tilde{q}_i + \tilde{\ell}_i - \bar{s}_i\|_{L^p(\omega_i)}^p \right)^{\frac{1}{p}} \leq \left( \frac{d^{d/2}}{2p+1} \right)^{\frac{1}{p}} \frac{d^2}{2m^2 N_2^2} |f|_{W^2_p(\omega)} .
\]
Consider
\[
s = \sum_{i=1}^{N_1} \bar{s}_i \chi_{\omega_i} .
\]
Then \( s \in S_2(P_m) \). Since \( m^{-3} \leq (mN_2)^{-2} \leq \left( \frac{|P_m|}{d} \right)^{-6/(2d+1)} \), we now combine (15) with (23) and (27) to deduce that
\[
\|f - s\|_p \leq \|f - \sum_{i=1}^{N_1} q_i \chi_{\omega_i}\|_p + \| \sum_{i=1}^{N_1} (q_i - \tilde{q}_i - \tilde{\ell}_i) \chi_{\omega_i}\|_p + \| \sum_{i=1}^{N_1} (\tilde{q}_i + \tilde{\ell}_i - \bar{s}_i) \chi_{\omega_i}\|_p \\
\leq C_1 |P_m|^{-6/(2d+1)} (|f|_{W^3_p(\alpha)} + |f|_{W^2_p(\alpha)}) ,
\]
where \( C_1 = d^{6/2+1} \left( d^2 (2 \rho_{d,3} + \rho_{d}^2) + \left( \frac{d^{d/2}}{2p+1} \right)^{\frac{1}{p}} \frac{d^2}{2} \right) \), and thereby proving (10).

For each \( i = 1, \ldots, N_1 \) we observe that
\[
|f - \bar{s}_i|_{W^2_p(\omega_i)} \leq 3^{1-\frac{n}{p}} \left( \sum_{\nu=1}^{d} \int_{\omega_i} \left( |D_{x^\mu} (f - q_i)(x)|^p \right. \right. \\
+ |D_{x^\mu} (q_i - \tilde{q}_i - \tilde{\ell}_i)(x)|^p + \left. \left. |D_{x^\mu} (\bar{s}_i)(x)|^p \right) dx \right)^{\frac{1}{p}} .
\]
For any \( \mu, \nu = 1, \ldots, d \), denoting by \( \sigma_{i,\nu}[\mu] \) the \( \mu \)-th coordinate of the eigenvector \( \sigma_{i,\nu} \), with \( |\sigma_{i,\nu}[\mu]| \leq 1 \), we again use the orthogonal change of variables
\[ y = \phi_{U_i}(x) \] and (25) to show that
\[ \|D_{x^\mu}(\lambda_{i,\nu} \ell^2_{\nu} - \bar{s}_{i,\nu})\|_{L^p(\omega_{ij}^{(\nu)})}^p = \int_{\omega_{ij}^{(\nu)}} 2|\lambda_{i,\nu} \sigma_{i,\nu}[\mu](\ell_{\nu}(x) - c_j)|^p \, dx \]
\[ \leq 2^p \int_{\phi_{U_i}(\omega_{ij}^{(\nu)})} |\lambda_{i,\nu}(y_{\nu} - c_j)|^p \, dy \]
\[ \leq 2^p \left( \frac{\sqrt{d}}{m} \right)^{d-1} \int_{c_j}^{c_j + \frac{w_{ij}}{mN_2}} |\lambda_{i,\nu}(y_{\nu} - c_j)|^p \, dy_{\nu} \]
\[ \leq \frac{2^p d^{d/2}}{(p + 1)m^d N_2} \left( \frac{\sqrt{d}|\lambda_{i,\nu}|}{mN_2} \right)^p \]
\[ \leq \frac{d^{d/2}}{(p + 1)N_2} \left( \frac{\sqrt{d}}{mN_2} \right)^p |f|_{W^2_p(\omega_i)}^p, \quad (29) \]
which implies
\[ \|D_{x^\mu}(\lambda_{i,\nu} \ell^2_{\nu} - \bar{s}_{i,\nu})\|_{L^p(\omega_i)}^p = \sum_{j=1}^{N_2} \|D_{x^\mu}(\lambda_{i,\nu} \ell^2_{\nu} - \bar{s}_{i,\nu})\|_{L^p(\omega_{ij}^{(\nu)})}^p \]
\[ \leq \frac{d^{d/2}}{p + 1} \left( \frac{\sqrt{d}}{mN_2} \right)^p |f|_{W^2_p(\omega_i)}^p. \]
Hence, for each \( \mu = 1, \ldots, d \), we have
\[ \|D_{x^\mu}(\bar{q}_i + \bar{\ell}_i - \bar{s}_i)\|_{L^p(\omega_i)}^p = \left\| \sum_{\nu=1}^{d} D_{x^\mu}(\lambda_{i,\nu} \ell^2_{\nu} - \bar{s}_{i,\nu}) \right\|_{L^p(\omega_i)}^p \]
\[ \leq \frac{d^{d/2}}{p + 1} \left( \frac{\sqrt{d}}{mN_2} \right)^p |f|_{W^2_p(\omega_i)}^p. \quad (30) \]
Combining (13), (24) and (30) shows that
\[ |f - \bar{s}|_{W^2_p(\omega_i)}^p \leq 3^{p-1} \left( \left( \frac{d\rho_{d,3}}{m^2} \right)^p |f|_{W^3_p(\omega_i)}^p + d^p (\rho_{d,3} + \rho_d^2)^p |f|_{W^3_p(\omega_i)}^p \right) + \frac{d^{d/2}}{p + 1} \left( \frac{d^{5/2}}{mN_2} \right)^p |f|_{W^2_p(\omega_i)}^p, \]
where, since \( m^{-2} \leq (mN_2)^{-1} \leq \left( \frac{|P_m|}{d} \right)^{-3/(2d+1)} \),
\[ |f - s|_{W^1_p(\Omega)} \leq C_2 |P_m|^{-3/(2d+1)} \left( |f|_{W^3_p(\Omega)} + |f|_{W^2_p(\Omega)} \right), \]
with \( C_2 = 3d^{d/2+1} (d\rho_{d,3} + d(\rho_{d,3} + \rho_d^2) + \frac{d^{d/2}}{p+1}) \), and (11) is proved. \( \square \)
4 Sums of piecewise linear polynomials with fixed directions

In the previous section, the splitting directions in Algorithm 3 depend on the eigenvectors of the average Hessian matrices of $f$. In this section, we present another method where the splitting directions are independent of the function.

**Lemma 2.** Any homogeneous quadratic polynomial $q$ can be represented as a linear combination of $\binom{d+1}{2}$ quadratic ridge functions

$$q = \sum_{\nu=1}^{d} a_{\nu} x_{\nu}^2 + \sum_{\nu=1}^{d-1} \sum_{\mu=\nu+1}^{d} b_{\nu\mu} (x_{\nu} + x_{\mu})^2,$$

where

$$a_{\nu} = \frac{1}{2} D_{x_{\nu}x_{\nu}} q - \frac{1}{2} \sum_{\mu \neq \nu} D_{x_{\nu}x_{\mu}} q, \quad b_{\nu\mu} = \frac{1}{2} D_{x_{\nu}x_{\mu}} q. \quad (32)$$

**Proof.** To prove the first statement we just need to find this representation for all quadratic monomials. For $q = x_{\nu}^2$, we simply take $a_{\nu} = 1$ and set all other coefficients to zero. Moreover, $2x_{\nu}x_{\mu} = (x_{\nu} + x_{\mu})^2 - x_{\nu}^2 - x_{\mu}^2$, so that for $q = x_{\nu}x_{\mu}$ with $\nu \neq \mu$ we can use $b_{\nu\mu} = \frac{1}{2}$, $a_{\nu} = a_{\mu} = -\frac{1}{2}$. The formulas (32) follow by a simple computation. \[\square\]

In the algorithm below, in contrast to Algorithm 3, the splitting directions of the macro-cells $\omega_i$ are independent of $f$.

**Algorithm 4.** Split $\Omega = (0,1)^d$ into $N_1 = m^d$, $m \in \mathbb{Z}_+$, cubes $\omega_1, \ldots, \omega_{N_1}$ of edge length $1/m$, whose edges are parallel to coordinate axes. For each $\nu = 1, \ldots, d$, define $\Delta^{(\nu)}$ by splitting each $\omega_i$ into $N_2$ slices $\omega^{(\nu)}_{ij}$, $j = 1, \ldots, N_2$, by equidistant hyperplanes orthogonal to the $x_{\nu}$-axis. For each pair $\{\nu, \mu\} \subseteq \{1, \ldots, d\}$, $\nu \neq \mu$, define $\Delta^{(\nu,\mu)}$ by splitting each $\omega_i$ into $N_2$ slices $\omega^{(\nu,\mu)}_{ij}$, $j = 1, \ldots, N_2$, by equidistant hyperplanes parallel to the subspace defined by $x_{\nu} + x_{\mu} = 0$. Set $\mathcal{P} = \{\Delta^{(1)}, \ldots, \Delta^{(d)}, \Delta^{(1,2)}, \ldots, \Delta^{(1,d)}, \ldots, \Delta^{(d-1,d)}\}$. Then $|\Delta^{(\nu)}| = |\Delta^{(\nu,\mu)}| = N_1 N_2$ for all $\nu, \mu = 1, \ldots, d$ and $|\mathcal{P}| = \binom{d+1}{2} N_1 N_2$.

Partitions $\Delta^{(1)}, \Delta^{(2)}$ and $\Delta^{(1,2)}$ in the case $d = 2$ and $N_2 = m = 4$ are illustrated in Figure 3.
Theorem 4. Let \( f \in W^3_p(\Omega) \), \( \Omega = (0,1)^d \), for some \( 1 \leq p \leq \infty \). For any \( m = 1,2, \ldots \), generate the system of partitions \( P_m \) by using Algorithm 4 with \( N_1 = m^d \) and \( N_2 = \lceil m^\frac{d}{3} \rceil \). Then there exists a sum of piecewise linear functions \( s_m \in S_2(P_m) \) such that

\[
E_2(f, P_m) \leq \| f - s_m \|_p \leq C_1|P_m|^{-6/(2d+1)}(|f|_W^3_p(\Omega) + |f|_W^3_p(\Omega)),
\]

\[
|f - s_m|_{W^1_p(\Omega)} \leq C_2|P_m|^{-3/(2d+1)}(|f|_W^3_p(\Omega) + |f|_W^3_p(\Omega)),
\]

where \( C_1, C_2 \) are constants depending only on \( d \).

Proof. As before we only consider the somewhat more difficult case when \( p < \infty \) and leave the modifications needed for \( p = \infty \) to the reader. Denote by \( \Delta_m \) the partition of \( \Omega \) into \( N_1 \) cubes \( \omega_1, \ldots, \omega_{N_1} \) of edge length \( 1/m \). For each \( i = 1, \ldots, N_1 \), from (3) that there exists a quadratic polynomial \( q_i \) such that

\[
\| f - q_i \|_{L_p(\omega_i)} \leq \rho_{d,3} \text{diam}(\omega_i)^3 |f|_{W^3_p(\omega_i)} \leq \frac{d^\frac{d}{2} \rho_{d,3}}{m^3} |f|_{W^3_p(\omega_i)},
\]

\[
|f - q_i|_{W^3_p(\omega_i)} \leq \rho_{d,3} \text{diam}(\omega_i)^2 |f|_{W^2_p(\omega_i)} \leq \frac{d \rho_{d,3}}{m^2} |f|_{W^2_p(\omega_i)},
\]

\[
|f - q_i|_{W^2_p(\omega_i)} \leq \rho_{d,3} \text{diam}(\omega_i) |f|_{W^1_p(\omega_i)} \leq \frac{\sqrt{d \rho_{d,3}}}{m} |f|_{W^1_p(\omega_i)}.
\]

By using (31) and the notation therein, let \( q_i = q_i^{(1)} + q_i^{(2)} \) where

\[
q_i^{(1)} = \sum_{\nu=1}^d a_\nu x_\nu^2, \quad \text{and} \quad q_i^{(2)} = \sum_{\nu=1}^d \sum_{\mu=\nu+1}^d b_{\nu\mu}(x_\nu + x_\mu)^2.
\]

For fixed \( \nu = 1, \ldots, d \) and \( j = 1, \ldots, N_2 \), there exists \( c_j \) such that the \( \nu \)-th side of \( \omega_{ij}^{(\nu)} \) is given by \([c_i, c_i + \frac{1}{mN_2}]\). Considering the linear polynomial
\[ s_i^{(1)} = \sum_{j=1}^{N_2} \sum_{\nu=1}^{d} (2a_{\nu}c_j x_{\nu} - a_{\nu}c_j^2) \chi_{\omega(j)^{\nu}}, \] clearly

\[
\|q_i^{(1)} - s_i^{(1)}\|_{L_p(\omega_i)} \leq d^{p-1} \sum_{j=1}^{N_2} \sum_{\nu=1}^{d} \int_{\omega_{ij}} |a_{\nu}|^p |x_{\nu} - c_j|^{2p} dx
\]

\[
= d^{p-1} \sum_{j=1}^{N_2} \sum_{\nu=1}^{d} \frac{1}{m^{d-1}} \int_{c_j}^{c_j + \frac{1}{mN_2}} |a_{\nu}|^p |x_{\nu} - c_j|^{2p} dx \\
= d^{p-1} \sum_{j=1}^{N_2} \sum_{\nu=1}^{d} \frac{1}{m^{d-1}} \int_{0}^{\frac{1}{mN_2}} |a_{\nu}|^p |x_{\nu}|^{2p} dx \\
= d^{p-1} \sum_{\nu=1}^{d} \frac{|a_{\nu}|^p}{(2p + 1)m^d} \left( \frac{1}{mN_2} \right)^{2p}.
\] (38)

By using the inequality between the arithmetic and the \( p \)-power means, together with (37), for each \( i = 1, \ldots, N_1 \), we have

\[
\sum_{\nu=1}^{d} \frac{|a_{\nu}|^p}{m^d} = \sum_{\nu=1}^{d} \int_{\omega_i} |\frac{1}{2} D_{x_{\nu}x_{\nu}, q_i}(x) - \frac{1}{2} \sum_{\mu \neq \nu} D_{x_{\nu}x_{\mu}, q_i}(x)|^p dx \\
\leq 2^{p-2} \sum_{\nu=1}^{d} \int_{\omega_i} \left( |D_{x_{\nu}x_{\nu}}(q_i - f)(x)|^p + \left| \sum_{\mu \neq \nu} D_{x_{\nu}x_{\mu}}(q_i - f)(x) \right|^p \right) dx \\
+ 2^p |f|^p_{W^{3}_p(\omega_i)} \\
\leq 2^p d^{p-1} \left( \frac{\sqrt{d} \rho_{d,3}}{m} \right)^{3p} |f|^p_{W^{3}_p(\omega_i)} + 2^p |f|^p_{W^{2}_p(\omega_i)}, \] (39)

and (38) becomes

\[
\|q_i^{(1)} - s_i^{(1)}\|_{L_p(\omega_i)} \leq \left( \frac{2d}{m^2 N_2^2} \right)^p \left( d^{p-1} \left( \frac{\sqrt{d} \rho_{d,3}}{m} \right)^{3p} |f|^p_{W^{3}_p(\omega_i)} + |f|^p_{W^{2}_p(\omega_i)} \right).
\] (40)

Given \( \nu = 1, \ldots, d \) and \( \mu = \nu + 1, \ldots, d \), there exists \( b_j \) such that the \( \nu \)-th side of \( \omega_{ij}^{(\nu, \mu)} \) lies between the hyperplanes \( x_{\nu} + x_{\mu} = b_j \) and \( x_{\nu} + x_{\mu} = b_j + w \) where \( 0 < w \leq \frac{\sqrt{d}}{mN_2} \). Consider the linear polynomial

\[
s_i^{(2)} = \sum_{j=1}^{N_2} \sum_{\nu=1}^{d} \sum_{\mu=\nu+1}^{d} 2b_j b_{\nu\mu} (x_{\nu} + x_{\mu} - b_j) \chi_{\omega_{ij}^{(\nu, \mu)}}.
\]

By using the change of variable \( X = x_{\nu} + x_{\mu} \) and \( Y = x_{\nu} - x_{\mu} \), where
\( b_j \leq X \leq b_j + w \) and the range of \( Y \) is at most \( \frac{\sqrt{d}}{m} \), we have

\[
\| q_i^{(2)} - s_i^{(2)} \|^p_{L^p(\omega)} \leq d^{2p-2} \left( \frac{1}{m^2 N_2} \right)^p \left( \left( \frac{\sqrt{d} \rho_{d,3}}{m} \right)^p |f|^{p}_{W^2_p(\omega)} + |f|^{p}_{W^2_{p}(\omega)} \right).
\]

(43)

With \( s_i = s_i^{(1)} + s_i^{(2)} \), combining (40) and (43) yields

\[
\| q_i - s_i \|^p_{L^p(\omega)} \leq 2^{p-1} \| q_i^{(1)} - s_i^{(1)} \|^p_{L^p(\omega)} + 2^{p-1} \| q_i^{(2)} - s_i^{(2)} \|^p_{L^p(\omega)}
\]

(44)

The inequality \( \max\{m^{-3}, (m N_2)^{-2}\} \leq 4^{(d+1)/2} \| p_m \|_{-6/(2d+1)} \) is easily provable. Considering \( s = \sum_{i=1}^{N_1} s_i \chi_{\omega_i} \) and \( q = \sum_{i=1}^{N_1} q_i \chi_{\omega_i} \), (35) and (44) imply

\[
\| f - s \|^p \leq \left( \sum_{i=1}^{N_1} \| f - q_i \|^p_{L^p(\omega)} \right)^{1/p} + \left( \sum_{i=1}^{N_1} \| q_i - s_i \|^p_{L^p(\omega)} \right)^{1/p}
\]

\[
\leq \frac{d^{3/2} \rho_{d,3}}{m^2} \| f \|_{W^2_p(\Omega)} + 4d^{3/2} \| f \|_{W^2_p(\Omega)} + 6d \left( \frac{\sqrt{d} \rho_{d,3}}{m N_2^2} + 1 \right) \left( |f|_{W^2_p(\Omega)} + |f|_{W^2_p(\Omega)} \right)
\]

\[
\leq C_1 \rho_{d,3} \| f \|_{W^2_p(\Omega)} + C_1 \rho_{d,3} |f|_{W^2_p(\Omega)}.
\]

(45)
where \( C_1 = 4^{(d+1)6/(2d+1)}(d^{2d} \rho_{d,3} + 10d^3(\sqrt{d} \rho_{d,3} + 1)) \), and the result (33) is proved.

For each \( i = 1, \ldots, N_1 \), by using the triangle inequality, we observe that

\[
|q_i - s_i|_{W^p_1(\omega_i)}^p \leq (2d)^{p-1} \sum_{\nu=1}^d \left( \|D_{x\nu}(q_i^{(1)} - s_i^{(1)})\|_{L^p(\omega_i)}^p + \|D_{x\nu}(q_i^{(2)} - s_i^{(2)})\|_{L^p(\omega_i)}^p \right).
\]

On one hand, a direct computation shows that, for each \( \nu = 1, \ldots, d \),

\[
\|D_{x\nu}(q_i^{(1)} - s_i^{(1)})\|_{L^p(\omega_i)}^p = \frac{2p}{p+1} |a_\nu|^p \left( \frac{1}{mN_2} \right)^p. \tag{45}
\]

On another hand, since for each \( k = 1, \ldots, d \),

\[
D_{x_k} \left( \sum_{\nu=1}^d \sum_{\mu=\nu+1}^d b_{\nu\mu} (x_\nu + x_\mu)^2 \right) = \sum_{\mu \neq k}^d 2b_{k\mu}(x_k + x_\mu),
\]

and

\[
D_{x_k} \left( \sum_{j=1}^{N_2} \sum_{\nu=1}^d \sum_{\mu=1}^d 2b_{\nu\mu} b_j (x_\nu + x_\mu) - b_{\nu\mu} b_j^2 \right) = \sum_{j=1}^{N_2} \sum_{\mu \neq k}^d 2b_{k\mu} b_j,
\]

we deduce that

\[
\|D_{x_k}(q_i^{(2)} - s_i^{(2)})\|_{L^p(\omega_i)}^p \leq d^{p-1} \sum_{j=1}^{N_2} \sum_{\mu \neq k}^d |2b_{k\mu}|^p \int_{\omega_i^{(\mu)}} |x_k + x_\mu - b_j|^p dx
\]

\[
\leq d^{p-1} \sum_{j=1}^{N_2} \sum_{\mu \neq k}^d |2b_{k\mu}|^p \left( \frac{\sqrt{d}}{m^{d-1}} \int_{b_j}^{b_j + \sqrt{d}} |X - b_j|^p dX \right)
\]

\[
= d^{p-1} \sum_{j=1}^{N_2} \sum_{\mu \neq k}^d |2b_{k\mu}|^p \frac{\sqrt{d}}{m^{d-1}} \frac{1}{p+1} \left( \frac{\sqrt{d}}{mN_2} \right)^{p+1}
\]

\[
= \frac{d^{p-1} 2p}{p+1} \sum_{\mu \neq k}^d \frac{|b_{k\mu}|^p}{m^{d-1}} \left( \frac{1}{mN_2} \right)^p. \tag{46}
\]

by virtue of a change of variable \( X = x_\nu + x_\mu \), \( Y = x_\nu - x_\mu \) where \( b_j \leq X \leq b_j + \frac{\sqrt{d}}{mN_2} \) and the range of \( Y \) not more that \( \frac{\sqrt{d}}{m} \). From (45) and (46), together with (39) and (42), we find that

\[
|q_i - s_i|_{W^p_1(\omega_i)}^p \leq \left( \frac{1}{mN_2} \right)^p \left( \frac{2p-1}{p+1} \right)^{\frac{p}{m}} \left( \frac{\sqrt{d} \rho_{d,3}}{m} \right)^p \|f\|_{W^p_1(\omega_i)}^p + \|f\|_{W^p_1(\omega_i)}^p.
\tag{47}
\]
It is easy to show that $m^{-2} \leq (mN_2)^{-1} \leq 2\binom{d+1}{2}^{3/(2d+1)}|\mathcal{P}_m|^{-3/(2d+1)}$. We deduce from (36) and (47) that

$$|f - s|_{W^p_p(\Omega)} \leq \left(2^{p-1} \sum_{i=1}^{N_1} \left(|f - q_i|_{W^p_p(\omega_i)}^p + |q_i - s_i|_{W^p_p(\omega_i)}^p\right)\right)^{\frac{1}{p}} \leq C_2 |\mathcal{P}_m|^{-3/(2d+1)} (|f|_{W^p_p(\Omega)} + |f|_{W^p_p(\Omega)}),$$

where $C_2 = 4\binom{d+1}{2}^{3/(2d+1)} \left(d^2 \rho_{d,3} + (d^2 + 2)\sqrt{d}\rho_{d,3} + 1\right)$, hence (34).

\section*{Acknowledgment}

Partial support by the Engineering and Physical Sciences Research Council of Great Britain under grant EP/G036136/1, Numerical Algorithms and Intelligent Software (NAIS) for the Evolving HPC Platform, is gratefully acknowledged. The work of Fabien Rabarison was supported in part by a PhD fellowship from the Scottish Funding Council.

\section*{References}


